NON-SPLIT EXTENSIONS OVER REPRESENTATION-FINITE HEREDITARY ALGEBRAS

Peter Brown

Abstract. Let $\Lambda = k\vec{Q}$ be the path algebra of a quiver with underlying graph A_n or D_n. For all pairs of indecomposable Λ-modules M and N, $\dim_k \operatorname{Ext}^1_\Lambda(N, M)$ is computed and a basis consisting of short exact sequences $0 \to M \to Y \to N \to 0$ is constructed.

Let Λ be a basic connected hereditary finite-dimensional algebra of finite representation type over an algebraically closed field k. In this paper, we investigate the indecomposable Λ-modules appearing as summands of the middle term of a non-split short exact sequence $0 \to M \to Y \to N \to 0$ where M and N are indecomposable Λ-modules.

According to Gabriel's theorem [5], a finite dimensional algebra Λ satisfying the above conditions is isomorphic to a path algebra $k\vec{Q}$ where the underlying graph Q of $\vec{Q}$ is one of the Dynkin diagrams A_n, D_n, E_6, E_7, or E_8. The Auslander-Reiten quiver of Λ, which will be denoted by Γ_Λ, is a finite connected translation subquiver of the infinite translation quiver $\mathbb{Z}Q$. Indecomposable Λ-modules will be identified with vertices of Γ_Λ.

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If $0 \to M \to Y \to N \to 0$ is a non-split short exact sequence, there are non-zero morphisms $M \to Y'$ and $Y' \to N$ for each indecomposable summand Y' of Y. These morphisms are sums of compositions of irreducible morphisms, which correspond to paths in Γ_Λ, so each indecomposable summand of Y lies on a path from M to N in Γ_Λ. Therefore, we consider the translation subquiver $\Gamma_\Lambda(M, N)$ of Γ_Λ whose vertices are the indecomposable Λ-modules lying on some path $M \to \cdots \to N$ in Γ_Λ. By the above considerations, all possible summands of the middle term of a non-split short exact sequence $0 \to M \to Y \to N \to 0$ are contained in $\Gamma_\Lambda(M, N)$.

$\operatorname{Ext}^1_\Lambda(N, M)$ will be viewed as the abelian group whose objects are isomorphism classes of short exact sequences $0 \to M \to Y \to N \to 0$, where addition is given by the Baer sum. $\operatorname{Ext}^1_\Lambda(N, M)$ is also a k-vector space via the actions of $k \cong (\operatorname{End}_\Lambda M)^{op} \cong \operatorname{End}_\Lambda N$. For each pair M, N of indecomposable Λ-modules where Λ is the path algebra of a quiver with underlying graph A_n or D_n, we will compute $\dim_k \operatorname{Ext}^1_\Lambda(N, M)$, then determine a k-basis for $\operatorname{Ext}^1_\Lambda(N, M)$, consisting of short exact sequences $0 \to M \to Y \to N \to 0$. The geometric configuration of $\Gamma_\Lambda(M, N)$ will be used to determine both the dimension, and the summands of the middle terms of basis elements.

Concepts of almost split sequences and the Auslander-Reiten quiver as described in [2] and [7] will be used. The Auslander-Reiten translate $D\operatorname{Tr}$ will denoted by τ, and $\operatorname{Tr}D$ will be denoted by τ^- (note that $\tau^{-k} = (\operatorname{Tr}D)^k$). Auslander proved in [1] that the almost split sequences generate the Grothendieck group $K_0(\mod \Lambda)$. Our constructions are based on expressing a non-split short exact sequence explicitly in terms of almost split sequences.

We will use the fact that when Λ is hereditary of finite representation type and M is an indecomposable Λ-module, the function $\dim_k \operatorname{Hom}_\Lambda(M, \quad) : \operatorname{ind} \Lambda \to \mathbb{Z}$ is the additive function (in the sense of Gabriel [6]) f_M associated with the vertex M in Γ_Λ. This follows because Γ_Λ is a preprojective component, hence it is standard (see [7] Lemma 2.2.3), that is, ind Λ is equivalent to the mesh category of Γ_Λ.

A path $M = M_0 \to M_1 \to \cdots \to M_t = X$ in Γ_Λ is called sectional if $M_{i-1} \neq \tau M_{i+1}$ for $1 \leq i \leq t-1$. The function f_M satisfies the properties that $f_M(X) = 1$ whenever there is a sectional path from M to X in Γ_Λ, and if $0 \to \tau X \to \bigoplus Y_i \to X \to 0$ is an almost split sequence, then $f_M(\tau X) + f_M(X) = \sum f_M(Y_i)$.

1. Intersections of sectional paths

In this section we consider a pair of sectional paths $M = M_0 \to M_1 \to \cdots M_{s-1} \to M_s = Z$, and $Z = N_0 \to N_1 \to \cdots N_{t-1} \to N_t = N$ in Γ_Λ, where $M_{s-1} = N_1$, and we determine conditions under which Z appears as a summand of the middle term of some non-split short exact sequence $0 \to M \to Y \to N \to 0$.

The first result in this direction applies to any artin algebra Λ.

Lemma 1.1. *Let Λ be an artin algebra.*

(a) Let $Z \xrightarrow{g} N$ be an irreducible morphism. There is a non-split short exact sequence $0 \to M \to Y \oplus Z \xrightarrow{(h,g)} N \to 0$ if and only if $\operatorname{Ext}^1_\Lambda(g, M) : \operatorname{Ext}^1_\Lambda(N, M) \to \operatorname{Ext}^1_\Lambda(Z, M)$ is not a monomorphism.

(b) Let $M \xrightarrow{f} Z$ be an irreducible morphism. There is a non-split short exact sequence $0 \to M \xrightarrow{k \oplus f} Y \oplus Z \to N \to 0$ if and only if $\operatorname{Ext}^1_\Lambda(N, f) : \operatorname{Ext}^1_\Lambda(N, M) \to \operatorname{Ext}^1_\Lambda(N, Z)$ is not a monomorphism.

Proof. We prove only (a) since the proof of (b) is dual. If $\operatorname{Ext}^1_\Lambda(g, M)$ is not mono, there is a pullback diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "g"] \arrow[dl, "h"'] & 0 \\
0 \arrow[r] & M \arrow[r, "u"] & Y \arrow[r, "v"] & N \arrow[r] & 0
\end{tikzcd}$$

where the lower sequence is non-split, but the top sequence splits. Then there exists a morphism $Z \xrightarrow{h} Y$ such that $g = vh$. Since g is irreducible, h must be a split mono since v is not a split epi, so Z is a summand of Y. Furthermore, $v|_Z = vh = g$, so v is of the form (h, g). For the converse, it is clear that the given sequence is a non-zero element of the kernel of $\operatorname{Ext}^1_\Lambda(g, M)$ (even if g is not irreducible). □

We may now identify one type of indecomposable Λ-module which appears as a summand of the middle term of a non-split short exact sequence $0 \to M \to Y \to N \to 0$.

Proposition 1.2. *Let Λ be hereditary of finite representation type, and let M and N be indecomposable Λ-modules. If there are sectional paths in Γ_Λ*

$$M = M_0 \to M_1 \to \cdots M_{s-1} \to M_s = Z$$

$$Z = N_0 \to N_1 \to \cdots N_{t-1} \to N_t = N$$

such that $M_{s-1} = \tau N_1$, and $\dim_k \operatorname{Ext}^1_\Lambda(N_j, M) = 1$ for $1 \leq j \leq t$, then Z is a summand of the middle term of the unique up to isomorphism non-split short exact sequence $0 \to M \to Y \to N \to 0$.

Proof. Since the paths are sectional, it follows from calculating dimensions via additive functions that $\operatorname{Ext}^1_\Lambda(Z, M) = \operatorname{Ext}^1_\Lambda(N, Z) = 0$. By Lemma 1.1, if $s = 1$ or $t = 1$ and $0 \to M \to Y \to N \to 0$ is non-split, then Z is a summand of Y.

Now assume that $s, t > 1$ and that the statement holds for the pair of sectional paths $M = M_0 \to M_1 \to \cdots M_{s-1} \to M_s = Z$, and $Z = N_0 \to N_1 \to \cdots N_{t-1}$, which have the required properties. Then there exists a non-split short exact sequence

$$(*) \qquad 0 \to M \xrightarrow{f \oplus g} Z \oplus Y \xrightarrow{(h,k)} N_{t-1} \to 0$$

By applying $\operatorname{Hom}_\Lambda(N, \quad)$ to this sequence, we obtain the exact sequence

$$0 = \operatorname{Hom}_\Lambda(N, N_{t-1}) \to \operatorname{Ext}^1_\Lambda(N, M) \to \operatorname{Ext}^1_\Lambda(N, Z \oplus Y) \to \operatorname{Ext}^1_\Lambda(N, N_{t-1}) = 0$$

where $\mathrm{Hom}_\Lambda(N, N_{t-1}) = 0$ because otherwise there would be an oriented cycle in Γ_Λ which is impossible since Λ is hereditary of finite representation type. Therefore

$$\mathrm{Ext}^1_\Lambda(N, Z \oplus Y) \cong \mathrm{Ext}^1_\Lambda(N, Y) \cong \mathrm{Ext}^1_\Lambda(N, M)$$

We have a commutative exact diagram

$$\begin{tikzcd}
 & (*) & (**) & & \\
 & 0 \arrow[d] & 0 \arrow[d] & & \\
\epsilon : 0 \arrow[r] & M \arrow[r] \arrow[d, "f\oplus g"'] & X \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
\eta : 0 \arrow[r] & Z \oplus Y \arrow[r] \arrow[d, "(h,k)"'] & Z \oplus X' \arrow[r] \arrow[d] & N \arrow[r] & 0 \\
 & N_{t-1} \arrow[r, equal] \arrow[d] & N_{t-1} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where η is the direct sum of a non-split sequence $0 \to Y \to X' \to N \to 0$, and $0 \to Z \overset{=}{\to} Z \to 0 \to 0$, and ϵ is the preimage of η under the isomorphism induced by $f \oplus g$. By applying $\mathrm{Hom}_\Lambda(N_{t-1}, \quad)$ to ϵ, we obtain

$$\mathrm{Hom}_\Lambda(N_{t-1}, X) \to \mathrm{Hom}_\Lambda(N_{t-1}, N) \to \mathrm{Ext}^1_\Lambda(N_{t-1}, M) \to \mathrm{Ext}^1_\Lambda(N_{t-1}, X) \to 0$$

Since Λ is hereditary of finite representation type, one of $\mathrm{Hom}_\Lambda(N_{t-1}, X)$ or $\mathrm{Ext}^1_\Lambda(N_{t-1}, X)$ must be zero. If $\mathrm{Hom}_\Lambda(N_{t-1}, X) = 0$, then $\dim_k \mathrm{Hom}_\Lambda(N_{t-1}, N) = \dim_k \mathrm{Ext}^1_\Lambda(N_{t-1}, M) = 1$ implies that $\mathrm{Ext}^1_\Lambda(N_{t-1}, X) = 0$. Therefore, $(**)$ splits, which shows that Z is a summand of X. $\square$

2. Exact sequences for A_n, D_n

Let $\Lambda \cong k\vec{Q}$ such that $\vec{Q}$ has underlying graph A_n or D_n. In this section we construct bases for $\mathrm{Ext}^1_\Lambda(N, M)$ for all pairs of indecomposable Λ-modules M, N such that $\mathrm{Ext}^1_\Lambda(N, M) \neq 0$. The following lemma, which is similar to Lemma 2.1 in [3], will be used in these constructions.

Lemma 2.1. *Let* Λ *be an artin algebra, and let*

$$0 \to A_1 \xrightarrow{f_1 \oplus g_1} B_1 \oplus A_2 \xrightarrow{(h_1, f_2)} B_2 \to 0$$

and

$$0 \to A_2 \xrightarrow{f_2 \oplus g_2} B_2 \oplus A_3 \xrightarrow{(h_2, f_3)} B_3 \to 0$$

be exact in $\mod \Lambda$. *Then the sequence*

$$0 \to A_1 \xrightarrow{f_1 \oplus g_2 g_1} B_1 \oplus A_3 \xrightarrow{(h_2 h_1, -f_3)} B_3 \to 0$$

is exact in $\mod \Lambda$. $\square$

The following statement describes the short exact sequences which will be constructed, based on the geometric configuration of Γ_Λ. The dimensions of $\operatorname{Ext}^1_\Lambda(N, M)$ are computed via additive functions. In the subquivers given in the statement of the theorem, vertices in parentheses may be zero. When two vertices are in angle brackets, at least one is non-zero.

Theorem 2.2. *Let* Λ *be the path algebra of a quiver with underlying graph* A_n *or* D_n*, and let* M *and* N *be indecomposable* Λ*-modules. Then* $\operatorname{Ext}^1_\Lambda(N, M) \neq 0$ *if and only if* $\Gamma_\Lambda(M, N)$ *is of one of the following configurations. If* $\Gamma_\Lambda(M, N)$ *is of one of these types, the corresponding sequences are exact, and they form a k-basis for* $\operatorname{Ext}^1_\Lambda(N, M)$.

$$\begin{tikzcd}
& & & & M_s \arrow[dr, "h_t"] & & & \\
& & & M_{s-1} \arrow[ur, "f_s"] \arrow[r, "f'_s"] \arrow[dr] & (M'_s) \arrow[r, "h'_t"] & \tau^- M_{s-1} \arrow[dr, "h_{t-1}"] & & \\
& & \cdots \arrow[ur, "f_{s-1}"] & & \tau^- M_{s-2} \arrow[ur] & & \cdots \arrow[dr, "h_1"] & \\
& M_1 \arrow[ur, "f_2"] & & & & & & N \\
M \arrow[ur, "f_1"] \arrow[dr, "g_1"] & & & & & & \cdots \arrow[ur, "k_1"] & \\
& \cdots \arrow[dr, dash, dotted] & & \tau^- N_{t-2} \arrow[dr] & & \cdots \arrow[ur, dash, dotted] & & \\
& & N_{t-1} \arrow[ur] \arrow[dr, "g_t"] & & \tau^- N_{t-1} \arrow[ur, "k_{s-1}"] & & & \\
& & & (N_t) \arrow[ur, "k_s"] & & & &
\end{tikzcd}$$

(1) $\dim_k \operatorname{Ext}^1_\Lambda(N, M) = 1$

$0 \to M \to M_s \oplus (M'_s) \oplus (N_t) \to N \to 0$

$$\begin{tikzcd}
M \arrow[dr, "u_1"] & & \tau^- M \arrow[dr] & & & & & & \tau^{-s+1} M \arrow[dr] & & \langle \tau^{-s} M \rangle \\
& M_1 \arrow[ur] \arrow[r] \arrow[dr, "u_2"] & E \arrow[r] & \tau^- M_1 & \cdots & \cdots & \cdots & \tau^{-s+2} M_1 \arrow[ur] \arrow[r] \arrow[dr] & E' \arrow[r] & \tau^{-s+1} M_1 \arrow[ur, "v_s"] \arrow[r] & \langle \tau^{s-1} E \rangle \\
& & M_2 \arrow[ur] \arrow[dr, "u_3"] & & & & & & \tau^{-s+2} M_2 \arrow[ur, "v_{s-1}"] & & \\
& & & \cdots \arrow[dr, "u_{s-1}"] & & & & \cdots \arrow[ur, "v_{s-2}"] & & & \\
& & & & M_{s-1} \arrow[dr, "u_s"] & & \tau^- M_{s-1} \arrow[ur, "v_2"] & & & & \\
& & & & & M_s \arrow[ur, "v_1"] & & & & &
\end{tikzcd}$$

(2) $\dim_k \operatorname{Ext}^1_\Lambda(N, M) = 1$ *(If s odd and $N = \tau^{-s} M$, or s even and $N = \tau^{s-1} E$)*

$0 \to M \to M_s \to N \to 0$

$$\begin{tikzcd}
M \arrow[dr, "u_1"'] & & \tau^- M \arrow[dr] & & & & & & \tau^{-s} M \arrow[dr, "h_t"] & & & & \\
 & M_1 \arrow[r] \arrow[ur] \arrow[dr, "u_2"'] & E \arrow[r] & \tau^- M_1 & \cdots & & \cdots & \tau^{-s+1} M_1 \arrow[ur, "v_s"] \arrow[r, "v'_s"] \arrow[dr] & E' \arrow[r, "h'_t"] & \tau^{-s} M_1 \arrow[dr, "h_{t-1}"] & & & \\
 & & M_2 \arrow[ur] \arrow[dr, "u_3"'] & & & & \tau^{-s+2} M_2 \arrow[ur, "v_{s-1}"] & & \tau^{-s+1} M_2 \arrow[ur] & & \tau^{-s} M_2 & & \\
 & & & \ddots \arrow[dr, "u_s"'] & & \iddots \arrow[ur, "v_{s-2}"] & & & & & & \ddots \arrow[dr, "h_1"] & \\
 & & & & M_s \arrow[ur, "v_1"] \arrow[dr, "g_1"'] & & & & & & & & N \\
 & & & & & N_1 \arrow[dr, "g_2"'] & & & & & & \iddots \arrow[ur, "k_1"'] & \\
 & & & & & & \ddots & & \tau^- N_{t-2} \arrow[dr] & & \iddots & & \\
 & & & & & & & N_{t-1} \arrow[ur] \arrow[dr, "g_t"'] & & \tau^- N_{t-1} \arrow[ur, "k_{s-1}"'] & & & \\
 & & & & & & & & (N_t) \arrow[ur, "k_s"'] & & & &
\end{tikzcd}$$

(3) $\dim_k \operatorname{Ext}^1_\Lambda(N,M) = 1$

$0 \to M \to \tau^{-s}M \oplus (N_t) \to N \to 0$, *if s is even*

$0 \to M \to E' \oplus (N_t) \to N \to 0$, *if s is odd*

$$\begin{tikzcd}
 & & & & & \tau^s N \arrow[dr, "h_s"] & & & & & & & & \tau N \arrow[dr] & & N \\
 & & & & \tau^s N_1 \arrow[ur, "f_t"] \arrow[r, "f'_t"] \arrow[dr] & E \arrow[r, "h'_s"] & \tau^{s-1} N_1 \arrow[dr, "h_{s-1}"] & & \cdots & & \cdots & & \tau N_1 \arrow[r] \arrow[ur] \arrow[dr] & E' \arrow[r] & N_1 \arrow[ur, "v_s"'] & \\
 & & & \iddots \arrow[ur, "f_{t-1}"] & & \tau^{s-1} N_2 \arrow[ur] & & \tau^{s-2} N_2 & & & & & & N_2 \arrow[ur, "v_{s-1}"'] & & \\
 & & \iddots & & & & & & \ddots & & & & \iddots & & & \\
 & \iddots & & & & & & & & \tau N_{s-1} \arrow[dr, "h_1"] & & N_{s-1} & & & & \\
M \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & & & & & & & & & N_s \arrow[ur, "v_1"'] & & & & & \\
 & \ddots & & & & & & & M_1 \arrow[urr, "k_1"'] & & & & & & & \\
 & & & \ddots \arrow[dr, "g_{s-1}"'] & & \tau M_{t-2} \arrow[dr] & & \iddots & & & & & & & & \\
 & & & & \tau M_{t-1} \arrow[ur] \arrow[dr, "g_s"'] & & M_{t-1} \arrow[ur, "k_{t-1}"'] & & & & & & & & & \\
 & & & & & (M_t) \arrow[ur, "k_t"'] & & & & & & & & & &
\end{tikzcd}$$

(4) $\dim_k \operatorname{Ext}^1_\Lambda(N,M) = 1$

$0 \to M \to \tau^s N \oplus (M_t) \to N \to 0$, *if s is even*

$0 \to M \to E \oplus (M_t) \to N \to 0$, *if s is odd*

$$\begin{tikzcd}[column sep=small]
 & & & M_s \arrow[dr, "h_t"] & & & & & & & & \tau^{-t}M_s \arrow[dr, "\eta_u"] & & & & \\
 & & M_{s-1} \arrow[ur, "f_s"] \arrow[r, "f'_s"] \arrow[dr] & M'_s \arrow[r, "h'_t"] & \tau^{-}M_{s-1} \arrow[dr, "h_{t-1}"] & & \cdots & \cdots & \cdots & & \tau^{-t}M_{s-1} \arrow[ur, "\alpha_t"] \arrow[r, "\alpha'_t"] \arrow[dr] & E \arrow[r, "\eta'_u"] & \tau^{-t-1}M_{s-1} \arrow[dr, "\eta_{u-1}"] & & & \\
 & \cdots \arrow[ur, "f_{s-1}"] & & \tau^{-}M_{s-2} \arrow[ur] & & \tau^{-2}M_{s-2} \arrow[dr, "h_{t-2}"] & & & & \tau^{-t}M_{s-2} \arrow[ur, "\alpha_{t-1}"] & & \tau^{-t-1}M_{s-2} \arrow[ur] & & \tau^{-t-2}M_{s-2} \arrow[dr] & & \\
M \arrow[ur, "f_1"] \arrow[dr, "g_1"'] & & & & & & \cdots \arrow[dr, "h_1"] & & \cdots \arrow[ur, "\alpha_{t-2}"] & & & & & & \cdots \arrow[dr, "\eta_1"] & \\
 & N_1 \arrow[dr, "g_2"'] & & & & & & X \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"'] & & & & & & & & N \\
 & & \cdots \arrow[dr, "g_{t-1}"'] & & & & \cdots \arrow[ur, "k_1"'] & & X_1 \arrow[dr, "\beta_2"'] & & & & & & \cdots \arrow[ur, "\nu_1"'] & \\
 & & & N_{t-1} \arrow[dr, "g_t"'] & & \cdots \arrow[ur] & & & & \cdots \arrow[dr] & & & & \cdots \arrow[ur] & & \\
 & & & & \langle N_t \rangle \arrow[ur, "k_s"'] \arrow[dr] & & & & & & X_{u-1} \arrow[dr, "\beta_u"'] & & \tau^{-}X_{u-1} \arrow[ur, "\nu_{t-1}"'] & & & \\
 & & & & & \cdots \arrow[dr] & & & & & & \langle X_u \rangle \arrow[ur, "\nu_t"'] & & & & \\
 & & & & & & \cdots \arrow[dr] & & \cdots \arrow[dr] & & \cdots \arrow[ur] & & & & & \\
 & & & & & & & \tau^{-v}X \arrow[ur] & (v>1) & X \arrow[ur] & & & & & &
\end{tikzcd}$$

(5) $\dim_k \operatorname{Ext}^1_\Lambda(N, M) = 1$

$$0 \to M \to \langle X_u \rangle \oplus \langle N_t \rangle \to N \to 0$$

$$\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & M_s \arrow[dr, "h_t"] & & & & & & & & \tau^{-t}M_s \arrow[dr, "\eta_u"] & & & & \\
 & & M_{s-1} \arrow[ur, "f_s"] \arrow[r, "f_s'"] \arrow[dr] & M_s' \arrow[r, "h_t'"] & \tau^{-}M_{s-1} \arrow[dr, "h_{t-1}"] & & \cdots & \cdots & \cdots & & \tau^{-t}M_{s-1} \arrow[ur, "\alpha_t"] \arrow[r, "\alpha_t'"] \arrow[dr] & E \arrow[r, "\eta_u'"] & \tau^{-t-1}M_{s-1} \arrow[dr, "\eta_{u-1}"] & & & \\
 & \iddots \arrow[ur, "f_{s-1}"] & & \tau^{-}M_{s-2} \arrow[ur] & & \tau^{-2}M_{s-2} \arrow[dr, "h_{t-2}"] & & & & \tau^{-t}M_{s-2} \arrow[ur, "\alpha_{t-1}"] & & \tau^{-t-1}M_{s-2} \arrow[ur] & & \tau^{-t-2}M_{s-2} & & \\
M \arrow[ur, "f_1"] \arrow[dr, "g_1"] & & & & & & \ddots \arrow[dr, "h_1"] & & \iddots \arrow[ur, "\alpha_{t-2}"] & & & & & & \ddots \arrow[dr, "\eta_1"] & \\
 & N_1 \arrow[dr, "g_2"] & & & & & & X \arrow[ur, "\alpha_1"] \arrow[dr, "\beta_1"] & & & & & & & & N \\
 & & \ddots \arrow[dr, "g_{t-1}"] & & & & \iddots \arrow[ur, "k_1"] & & X_1 \arrow[dr, "\beta_2"] & & & & & & \iddots \arrow[ur, "\nu_1"] & \\
 & & & N_{t-1} \arrow[dr, "g_t"] & & \iddots & & & & \ddots & & & & \iddots & & \\
 & & & & N_t \arrow[ur, "k_s"] \arrow[dr, "\rho_1"] & & & & & & X_{u-1} \arrow[dr, "\beta_u"] & & \tau^{-}X_{u-1} \arrow[ur, "\nu_{t-1}"] & & & \\
 & & & & & Y_1 & & & & & & X_u \arrow[ur, "\nu_t"] & & & & \\
 & & & & & & \ddots \arrow[dr, "\rho_{u-1}"] & & \tau^{-}Y_{u-2} \arrow[dr] & & \iddots \arrow[ur, "\sigma_1"] & & & & & \\
 & & & & & & & Y_{u-1} \arrow[ur] \arrow[dr, "\rho_u"] & & \tau^{-}Y_{u-1} \arrow[ur, "\sigma_{s-1}"] & & & & & & \\
 & & & & & & & & (Y_u) \arrow[ur, "\sigma_s"] & & & & & & &
\end{tikzcd}$$

(6) $\dim_k \operatorname{Ext}^1_\Lambda(N, M) = 2$

$$\textit{If } t \textit{ is even, } \begin{cases} 0 \to M \to M_s \oplus E \oplus (Y_u) \to N \to 0 \\ \\ 0 \to M \to M_s' \oplus \tau^{-t}M_s \oplus (Y_u) \to N \to 0 \end{cases}$$

$$\textit{If } t \textit{ is odd, } \begin{cases} 0 \to M \to M_s \oplus \tau^{-t}M_s \oplus (Y_u) \to N \to 0 \\ \\ 0 \to M \to M_s' \oplus \ E \oplus (Y_u) \to N \to 0 \end{cases}$$

Proof. In case (1), $\dim_k \operatorname{Ext}^1_\Lambda(N, M) = 1$, there is up to isomorphism a unique non-split short exact sequence $0 \to M \to Y \to N \to 0$. By Proposition 1.2, M_s is a summand of Y, and so are M_s' and N_t when they are non-zero. By induction on s and t, the length of Y is equal to the sum of the lengths of M_s, M_s' and N_t, so $Y = M_s \oplus (M_s') \oplus (N_t)$.

For the cases in (2), it is again easily shown that the length of M_s equals the sum of the lengths of M and N. Since M_s must be a summand of the unique non-split short exact sequence $0 \to M \to Y \to N \to 0$ by Proposition 1.2, it again follows that $Y \cong M_s$.

(3) The sequence $0 \to M_s \to \tau^{-s}M \oplus E' \oplus (N_t) \to N \to 0$ is exact by (1). If s is even, we have an exact sequence $0 \to M \to M_s \to E' \to 0$, and if s is odd, we have an exact sequence $0 \to M \to M_s \to \tau^{-s}M \to 0$. Since there are sectional paths from M_s to $\tau^{-s}M$ and E', $\dim_k \operatorname{Hom}_\Lambda(M_s, \tau^{-s}M) = \dim_k \operatorname{Hom}_\Lambda(M_s, E') = 1$, so we may assume that the maps $M_s \to \tau^{-s}M$ and $M_s \to E'$ appearing in the sequences are the same maps. Then from Lemma 1.2, we obtain the short exact sequences $0 \to M \to \tau^{-s}M \oplus (N_t) \to N \to 0$ if s is even, and $0 \to M \to E' \oplus (N_t) \to N \to 0$ if s is odd. The sequences in (4) are obtained similarly.

(5) First assume that t is even. Then from (3) and (4) we have non-split short exact sequences $0 \to M \to M_s \oplus \langle N_t \rangle \to \tau^{-t}M_s \to 0$ and $0 \to M_s \to \tau^{-t}M_s \oplus \langle X_u \rangle \to N \to 0$. Since t is even, $\dim_k \operatorname{Hom}_\Lambda(M_s, \tau^{-t}M_s) = 1$ so we may assume that the maps $M_s \to \tau^{-t}M_s$ appearing in the two sequences are the same map. Now, by Lemma 1.2, we obtain the exact sequence $0 \to M \to \langle X_u \rangle \oplus \langle N_t \rangle \to N \to 0$. The case in which t is odd is handled similarly.

(6) We obtain only the first sequence in the case where t is even, since the other cases are similar. By (1) we have a short exact sequence $0 \to M \to M_s \oplus M'_s \oplus (Y_u) \to X_u \to 0$, and by (3), we have a short exact sequence $0 \to M'_s \to E \oplus X_u \to N \to 0$. Since $\dim_k \operatorname{Hom}_\Lambda(M'_s, X_u) = 1$ we may again combine the sequences to obtain $0 \to M \to M_s \oplus E \oplus (Y_u) \to N \to 0$. $\square$

Remark: When $\Gamma_\Lambda(M,N)$ is of type (6), there is an exact sequence of the form $0 \to M \to X_u \oplus N_t \to N \to 0$ by the same argument given for (5), so this sequence is a linear combination of the two sequences given as basis elements.

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DÉPARTEMENT DE MATHÉMATIQUES ET D'INFORMATIQUE, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE (QUÉBEC) J1K 2R1 CANADA

CURRENT ADDRESS: ARTIOS CORPORATION 40 WESTOVER RD. LUDLOW, MA 01056-1200 USA
E-mail address: `pbrown@math.umass.edu`