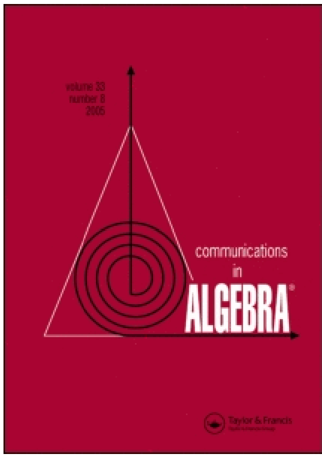

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GLOBAL DIMENSION OF QUASIHEREDITARY SERIAL RINGS

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0. Introduction.

Let A be an indecomposable artin ring with Jacobson radical $\underline{\mathbf{r}}$, and let $P_1, \ldots, P_n$ be a complete set of non-isomorphic indecomposable projective A-modules. We prove that if A is quasihereditary and serial then $\operatorname{gldim} A \leq \lambda(A) + 1 \leq n$ where $\lambda(A)$ is defined to be the number of simple A-modules S with $\operatorname{pd}_A S \neq 1$. Recall that an artin ring is serial if its indecomposable projective and injective modules are uniserial; that is, have a unique composition series. All indecomposable modules over a serial ring are uniserial. The Loewy length of an A-module M, denoted $\ell\ell(M)$, is the least integer k such that $\underline{\mathbf{r}}^k M = 0$. M is uniserial if and only if $\ell(M) = \ell\ell(M)$ where $\ell(M)$ is the composition length of M. See [**EG**] for more information on serial rings.

An heredity ideal J of A is an ideal such that (a) $J^2 = J$, i.e. J is an idempotent ideal; (b) $J\underline{\mathbf{r}}J = 0$; and (c) J is projective as a left A-module. Burgess and Fuller prove in [**BF**] that a serial ring is quasihereditary if and only if it has an heredity ideal. It is known that quasihereditary rings are of finite global dimension. From results of Dlab and Ringel [**DR**] and Gustafson [**G**] it is known that if a ring is either quasihereditary or serial of finite global dimension then $\operatorname{gldim} A \leq 2n - 2$.

We will use the fact that if A is serial, then it has at most one simple projective module, and the fact (see [**AF**], [**F**]) that the non-isomorphic indecomposable projective A-modules can be ordered $P_1, \ldots, P_n$ such that $P_{i+1} \to \underline{\mathbf{r}}P_i$ is a projective cover for $i = 1, \ldots, n-1$ and $P_1 \to \underline{\mathbf{r}}P_n$ is a projective cover if $\underline{\mathbf{r}}P_n \neq 0$ (such an ordering is called a Kupisch series). The category of finitely generated left A-modules will be denoted $\operatorname{mod} A$.

1. Preliminaries.

The proof of the bound on global dimension will proceed by induction on $\lambda(A)$. If P is an indecomposable projective A-module of minimal length, then $\underline{\mathbf{r}}P$ cannot be a nonzero projective, so $\operatorname{pd}_A P/\underline{\mathbf{r}}P \neq 1$. Therefore $\lambda(A) \geq 1$ for every A. The following result will be the basis for the induction.

Lemma 1.1. *Let A be an artin ring with $\lambda(A) = 1$. Then there is a unique indecomposable projective P of minimal length, and*

a) $\operatorname{gldim} A = 1$ if P is simple.

b) $\operatorname{gldim} A = 2$ if P is not simple and $\operatorname{End}_A P$ is a division ring.

c) $\operatorname{gldim} A = \infty$ if $\operatorname{End}_A P$ is not a division ring.

Proof: P is unique because the simple top of any indecomposable projective module of minimal length has projective dimension not equal to one as noted above, and by assumption, there is only one such simple module. If P is simple, then (a) follows because all simple modules then have projective dimension at most one. For (b), P is not simple so $\underline{\mathbf{r}}P \neq 0$. Since $\operatorname{End}_A P$ is a division ring, $\operatorname{rad}\operatorname{End}_A P = \operatorname{Hom}_A(P, \underline{\mathbf{r}}P) = 0$ so $P/\underline{\mathbf{r}}P$ is not a composition factor of $\underline{\mathbf{r}}P$. Therefore, $\operatorname{pd}_A \underline{\mathbf{r}}P = 1$ and $\operatorname{pd} P/\underline{\mathbf{r}}P = 2$ so $\operatorname{gldim} A = 2$.

The proof of (c) is patterned after the proof of Lemma 2 in [**Z**]: Suppose $P_1, \ldots, P_n$ are the non-isomorphic indecomposable projective A-modules where P_n is the unique indecomposable projective A-module of minimal length. Let $B = (\operatorname{End}_A P_n)^{op}$. B is local but not a division ring so $\operatorname{gldim} B = \infty$. The functor $\operatorname{Hom}_A(P_n, \quad) : \operatorname{mod} A \to \operatorname{mod} B$ is exact. We claim that $\operatorname{Hom}_A(P_n, P_i)$ is B-projective for all $i = 1, \ldots, n$. The proof is by induction on the length of P_i. If

$\ell(P_i) = \ell(P_n)$ then $P_i = P_n$ since P_n is the unique indecomposable projective of minimal length, so $\mathrm{Hom}_A(P_n, P_i)$ is projective over B. Assume $\ell(P_i) > \ell(P_n)$ and that $\mathrm{Hom}_A(P_n, P_j)$ is projective over B is $\ell(P_j) < \ell(P_i)$. . Then $\underline{\mathrm{r}}P_i$ is projective over A, so $\underline{\mathrm{r}}P_i = \coprod_{\ell(P_j)<\ell(P_i)} a_j P_j$, where $a_j \geq 0$. Now

$$\begin{aligned}\mathrm{Hom}_A(P_n, P_i) &= \mathrm{Hom}_A(P_n, \underline{\mathrm{r}}P_i)\\ &= \mathrm{Hom}_A(P_n, \coprod_{\ell(P_j)<\ell(P_i)} a_j P_j)\\ &= \coprod_{\ell(P_j)<\ell(P_i)} a_j \mathrm{Hom}_A(P_n, P_j)\end{aligned}$$

which is projective over B by induction.

Now, applying $\mathrm{Hom}_A(P_n, \quad)$ to a minimal projective resolution of $S_n = P_n/\underline{\mathrm{r}}P_n$ over A yields a projective resolution over B of $\mathrm{Hom}_A(P_n, S_n)$, which is the unique simple B-module. Since the latter resolution is infinite, it follows that $\mathrm{pd}_A S_n = \infty$, so gldim $A = \infty$. □

We will need the following result concerning projective modules over a serial ring.

Lemma 1.2. *Let A be serial, and let P be a projective A-module.*

a) If Y is an indecomposable A-module with $P \subset Y$, then Y is projective.

b) Let $0 \to Y \xrightarrow{s} P' \xrightarrow{t} X$ be exact where $P \subset P'$, P' is indecomposable and $\mathrm{Hom}_A(P, X) = 0$. Then $P \subset Y$ and Y is projective.

Proof: (a) Consider the commutative diagram with exact row

$$\begin{array}{ccccc} & & P & & \\ & \overset{g}{\swarrow} & \downarrow i & & \\ Q & \xrightarrow{k} & Y & \to & 0 \end{array}$$

where i is the inclusion map, $Q \xrightarrow{k} Y$ is a projective cover and g is a map such that $kg = i$ (g exists because P is projective). Q is uniserial, so its submodules are totally ordered. Therefore $\mathrm{Ker}\, k \subset \mathrm{Im}\, g$, since $\mathrm{Im}\, g \subset \mathrm{Ker}\, k$ implies $i = 0$. Now $kg = i$ is a monomorphism, so k is monic on $\mathrm{Im}\, g$ which contains $\mathrm{Ker}\, k$ so $\mathrm{Ker}\, k = 0$. Hence k is an isomorphism, and Y is projective.

(b) The composition $P \xrightarrow{i} P' \xrightarrow{t} X$ is zero where i is the inclusion map, so from the universal property of kernels there exists a morphism $P \xrightarrow{u} Y$ such that the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & Y & \xrightarrow{s} & P' & \xrightarrow{t} & X \\ & & & \nwarrow_{u} & \uparrow i & & \\ & & & & P & & \end{array}$$

commutes. Now $i = su$ is mono, so u is mono, hence $P \subset Y$. Since Y is indecomposable, it is projective by (a). □

If X and Y are A-modules, let $\tau_X(Y)$ denote the trace of X in Y, that is the submodule of Y generated by the images of all homomorphisms $X \rightarrow Y$. For any A-module M, $\tau_M(A)$ is an ideal of A. If A is a quasihereditary ring, it has an heredity ideal of the form $\tau_P(A)$ where P is an indecomposable projective module with the property that for every projective A-module Q, every non-zero homomorphism $P \rightarrow Q$ is a monomorphism [**BF**, Lemma 1.7]. Such a projective module will be called ℓ-projective. If P is ℓ-projective, every non-zero map from P to itself is a monomorphism, hence an isomorphism, so $\mathrm{End}_A P$ is a division ring. This means that $P/\underline{\mathbf{r}}P$ is not a composition factor of $\underline{\mathbf{r}}P$, so if A is serial, then $\ell\ell(P) \leq n$. Clearly any simple projective is ℓ-projective. The following lemma guarantees the existence of an ℓ-projective with non-projective radical.

Lemma 1.3. *Let A be a serial ring, and let P be ℓ-projective. If $\underline{\mathbf{r}}P$ is projective, then $\underline{\mathbf{r}}P$ is ℓ-projective.*

Proof: If $\underline{\mathbf{r}}P$ is simple we are done. Otherwise, it suffices to show that any non-zero map $\underline{\mathbf{r}}P \xrightarrow{h} Q$ where Q an indecomposable projective is a monomorphism. Assume first that $Q \cong \underline{\mathbf{r}}P$. Since P is ℓ-projective, $\ell\ell(P) \leq n$, so $\ell\ell(\underline{\mathbf{r}}P) \leq n-1$, which implies that $\mathrm{End}_A \underline{\mathbf{r}}P$ is a division ring. In particular, every morphism $\underline{\mathbf{r}}P \rightarrow \underline{\mathbf{r}}P$ is a monomorphism.

Assume that $Q \ncong \underline{\mathbf{r}}P$. Since Q is uniserial, $\mathrm{Im}\, h = \underline{\mathbf{r}}^i Q$ for some integer i. Then $\underline{\mathbf{r}}P/\underline{\mathbf{r}}^2 P \cong \underline{\mathbf{r}}^i Q/\underline{\mathbf{r}}^{i+1} Q$. But $Q \ncong \underline{\mathbf{r}}P$ implies $i \geq 1$, so $\underline{\mathbf{r}}^{i-1} Q/\underline{\mathbf{r}}^i Q \cong P/\underline{\mathbf{r}}P$. This means that there is a non-zero map $P \rightarrow Q$ which then must be a monomorphism, so $\underline{\mathbf{r}}^{i-1} Q \cong P$, hence $\mathrm{Im}\, h \cong \underline{\mathbf{r}}^i Q \cong \underline{\mathbf{r}}P$ which implies h is a monomorphism. □

From the above result, it follows that if A is quasihereditary serial, then there is an ℓ-projective A-module P such that $\mathrm{pd}_A P/\underline{\mathrm{r}}P \neq 1$. In the following, A and A/J are compared, where J is the trace ideal of a certain ℓ-projective A-module.

Lemma 1.4. *Let A be a quasihereditary serial ring, and let $P_1, \dots, P_n$ be a complete set of non-isomorphic indecomposable projective A-modules. Let P be an ℓ-projective A-module such that $\mathrm{pd}_A P/\underline{\mathrm{r}}P \neq 1$ and such that P is simple if there is a simple projective A-module. Let $J = \tau_P(A)$. Then*

a) A/J has $n-1$ non-isomorphic simple modules.

b) $\lambda(A) - 1 \leq \lambda(A/J) \leq \lambda(A)$

c) A/J has a simple projective.

Proof: We may assume that $P = P_n$, and that $\underline{\mathrm{r}}P_k/\underline{\mathrm{r}}^2 P_k = P_{k+1}/\underline{\mathrm{r}}P_{k+1}$, $1 \leq k \leq n-1$ where $\underline{\mathrm{r}}P_k \neq 0$ for $1 \leq k \leq n-1$ since P_n is the only possible simple projective. Now (a) follows from the fact that $P_1/\tau_P(P_1), \dots, P_{n-1}/\tau_P(P_{n-1})$ are the indecomposable non-isomorphic projective A/J modules.

Let $\tilde{P} = I(P)$, be the injective envelope of P. By Lemma 1.2 (a), $\tilde{P}$ is projective, and every submodule of $\tilde{P}$ containing P is projective. Any indecomposable projective module containing P has injective envelope $\tilde{P}$, and is therefore a submodule of $\tilde{P}$. Now $\tilde{P} = P_K$ for some $K \leq n-1$ because $P = \tilde{P}$ implies that P_{n-1} is simple, which contradicts our assumption.

By the ordering chosen, there is an epimorphism $P_{K+1} \xrightarrow{e} \underline{\mathrm{r}}P_K \to 0$, and since $P \subset P_K$, there is a commutative diagram

$$\begin{array}{ccccc} & & P & & \\ & \overset{s}{\swarrow} & \downarrow i & & \\ P_{K+1} & \xrightarrow{e} & \underline{\mathrm{r}}P_K & \to & 0 \end{array}$$

where s is a monomorphism because P is ℓ-projective. Now $i = es$ is also a monomorphism, so e is a monomorphism, hence an isomorphism, so by induction, for $i = K, \dots, n-1$, $P_{i+1} \cong \underline{\mathrm{r}}P_i$.

Let $S_i = P_i/\underline{\mathrm{r}}P_i$ for each $i = 1, \dots, n$. If $K \leq i \leq n-1$, then there is an exact sequence $0 \to P_{i+1} \to P_i \to S_i \to 0$, so $\mathrm{pd}_A S_i = 1$. Since $P \subset P_i, P_{i+1}$, and

$\tau_P(P_i) \cong \tau_P(P_{i+1}) \cong P$, there is a commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & P & = & P & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & P_{i+1} & \to & P_i & \to & S_i & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & P_{i+1}/\tau_P(P_{i+1}) & \to & P_i/\tau_P(P_i) & \to & S_i & \to & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

with the bottom row being a projective resolution for S_i over A/J, so $\text{pd}_{A/J}S_i = 1$, for $i = K, \dots, n-1$. Since $P_n \to \underline{\text{r}}P_{n-1}$ is non-zero, it is mono so $P_{n-1}/\tau_P(P_{n-1})$ is simple (which proves (c)) and its projective dimension over A/J is 0. Therefore S_{n-1} is the only simple A/J-module in $S_K, \dots, S_{n-1}$ with $\text{pd}_{A/J}S_i \neq 1$, and S_n is the only simple A-module in $S_K, \dots, S_n$ with $\text{pd}_A S_i \neq 1$ by the way the ℓ-projective $P = P_n$ was chosen.

For $i = 1, \dots, K-1$, $\text{Hom}_A(P, P_i) = 0$, so $P_i/\tau_P(P_i) = P_i$ is projective over A/J. Therefore, for $i = 1, \dots, K-2$, if $S_i = P_i/\underline{\text{r}}P_i$ has $\text{pd}_A S_i = 1$, then $\text{pd}_{A/J}S_i = 1$, so the number of simple modules among $S_1, \dots, S_{K-2}$ of projective dimension not equal to 1 is the same over A and A/J. The only simple module left to consider is S_{K-1}. By the choice of K, $\text{pd}_A S_{K-1} \neq 1$. If $P_K/\tau_P(P_K) = \underline{\text{r}}P_{K-1}$ then $\text{pd}_{A/J}S_{K-1} = 1$, so in this case $\lambda(A/J) = \lambda(A) - 1$. Otherwise, $\lambda(A/J) = \lambda(A)$. ☐

It is known in general that $\lambda(A) \leq n-1$ if A is serial of finite global dimension. However, using the notation of the proof of Lemma 1.4, $P_K/\underline{\text{r}}P_K$ is a simple module of projective dimension 1. Also note that A/J is again quasihereditary serial because if P' is the simple A/J-module in (c), then $\tau_{P'}(A/J)$ is an heredity ideal. The quotient of a serial ring is clearly a serial ring.

In the following we further investigate what $\lambda(A/J) = \lambda(A)$ means.

Lemma 1.5. *Let A be a quasihereditary serial ring. Let P and J be as in Lemma 1.4. Suppose $0 \to Q' \to Q'' \to X \to 0$ is exact with $\text{Hom}_A(P, X) = 0$ and $P \subsetneq Q''$. If $\lambda(A/J) = \lambda(A)$, then $P \subsetneq Q'$.*

Proof: Let $P_1, \dots, P_{K-1}, P_K, \dots, P_n$ be as in the proof of Lemma 1.4. The kernel of the epimorphism $P_K \to \underline{\text{r}}P_{K-1} \to 0$ contains P by Lemma 1.2 (b), so we have an induced epimorphism $P_K/\tau_P(P_K) \to \underline{\mathbf{r}}P_{K-1} \to 0$, which is a proper epimorphism when $\lambda(A/J) = \lambda(A)$. The composition factors of $P_K/\tau_P(P_K)$ are $S_K, S_{K+1}, \ldots, S_{n-1}$, so a proper quotient does not contain $\operatorname{soc} P_K/\tau_P(P_K) = S_{n-1}$ as a composition factor. Therefore, $\underline{\mathbf{r}}P_{K-1}$ does not have S_{n-1} as a composition factor, hence $\operatorname{Hom}_A(P_{n-1}, P_{K-1}) = 0$.

We claim that $\operatorname{Hom}_A(P_{n-1}, X) = 0$. If $P_{n-1} \xrightarrow{h} P_{K-2}$ is a map, then h is not an isomorphism so its image is contained in $\underline{\mathbf{r}}P_{K-2}$. Thus we have a commutative diagram

$$\begin{array}{ccccc} & & P_{n-1} & & \\ & {}^{v}\swarrow & \downarrow h & & \\ P_{K-1} & \xrightarrow{p} & \underline{\mathbf{r}}P_{K-2} & \to & 0 \end{array}$$

but $\operatorname{Hom}_A(P_{n-1}, P_{K-1}) = 0$ implies $h = pv = 0$. By induction, $\operatorname{Hom}_A(P_{n-1}, P_i) = 0$ for $i = 1, \ldots, K-1$, so the claim follows.

From Lemma 1.2, it follows that $P \subset Q'$ and Q' is projective. If $P = Q'$ in the above exact sequence, this implies that $X \cong Q''/P$, so $\operatorname{soc} X = \operatorname{soc} Q''/P = S_{n-1}$. Therefore $\operatorname{Hom}_A(P_{n-1}, X) \neq 0$, contradicting the claim, so $P \subsetneq Q'$. □

2. Computing Global Dimension.

In this section, the global dimension of a quasihereditary serial ring A is computed. This will be done by comparing projective resolutions of simple modules over A and A/J.

Lemma 2.1. *Let A be a quasihereditary serial ring with P ℓ-projective such that $\operatorname{pd}_A P/\underline{\mathbf{r}}P \neq 1$ and such that P is simple if there is a simple projective A-module. Let $J = \tau_P(A)$. If $\lambda(A/J) = \lambda(A)$, then for every simple module S with $S \not\cong P/\underline{\mathbf{r}}P$, S is a simple A/J-module, and $\operatorname{pd}_A S = \operatorname{pd}_{A/J} S$.*

Proof: A is quasihereditary so its global dimension is finite. Let $0 \to Q_s \xrightarrow{f_s} Q_{s-1} \xrightarrow{f_{s-1}} \cdots \to Q_0 \xrightarrow{f_0} S \to 0$ be a projective resolution over A of S where $S \not\cong P/\underline{\mathbf{r}}P$. $\operatorname{Hom}_A(P, S) = 0$, so S is a simple A/J-module. If $\operatorname{Hom}_A(P, Q_i) = 0$ for $i = 0, \ldots, s$ then each Q_i is an A/J-module, and therefore $\operatorname{pd}_A S = \operatorname{pd}_{A/J} S$.

Suppose $P \subset Q_i$ for some i. Since $\operatorname{Hom}_A(P, S) = \operatorname{Hom}_A(P, \operatorname{Im} f_0) = 0$, there exists a $t > 0$ minimal such that $P \subset Q_t$ and $\operatorname{Hom}_A(P, \operatorname{Im} f_t) = 0$. If $Q_t = P$

then $Q_t \xrightarrow{f_t} Q_{t-1}$ is a monomorphism, so $P \subset Q_{t-1}$, contrary to minimality of t, therefore $P \subsetneq Q_t$. By Lemma 1.2, the sequence $0 \to Y \to Q_t \xrightarrow{f_t} \operatorname{Im} f_t \to 0$ has Y projective, so $t = s-1$, $Y = Q_s$, and $\operatorname{Hom}_A(P, Q_i) = 0$ for all $i = 0, \ldots, s-2$. Now the sequence

$$0 \to \operatorname{Im} f_{s-1} \to Q_{s-2} \xrightarrow{f_{s-2}} \cdots \to Q_0 \xrightarrow{f_0} S \to 0$$

is exact over A/J, and each $Q_0, \ldots, Q_{s-2}$ is projective over A/J. Since $\lambda(A/J) = \lambda(A)$, we have by Lemma 1.5 that $P \subsetneq Q_s$. Since $P \cong \tau_P(Q_s) \cong \tau_P(Q_{s-1})$ and $Q_s/\tau_P(Q_s)$,
$Q_{s-1}/\tau_P(Q_{s-1}) \neq 0$, there is a commutative exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & Q_s & \to & Q_s/\tau_P(Q_s) & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & P & \to & Q_{s-1} & \to & Q_{s-1}/\tau_P(Q_{s-1}) & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \operatorname{Im} f_{s-1} & = & \operatorname{Im} f_{s-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where the last column forms a minimal A/J-projective resolution of $\operatorname{Im} f_{s-1}$, so combining this with the above exact sequence, we get $\operatorname{pd}_A S = \operatorname{pd}_{A/J} S$. □

In [**DR**], it is proved that if A is a semiprimary ring and J is an ideal of A such that J is projective and $J\underline{\mathbf{r}}J = 0$, then $\operatorname{gldim} A \leq \operatorname{gldim} A/J + 2$. In particular, this holds when J is an heredity ideal. The following result shows that the global dimensions differ by at most one if A is serial with a simple projective module.

Lemma 2.2. *If A is a serial ring with P a simple projective module and $J = \tau_P(A)$ then* $\operatorname{gldim} A \leq \operatorname{gldim} A/J + 1$.

Proof: As in Lemma 2.1, a projective resolution over A is either a resolution over A/J or only the last two terms contain P. If P appears in the resolution, it must be the last term since it is simple. As in the proof of Lemma 2.1 we obtain a projective resolution $0 \to Q_s/\tau_P(Q_s) \to Q_{s-1}/\tau_P(Q_{s-1}) \to Q_{s-2} \to \cdots \to Q_0 \xrightarrow{f_0} S \to 0$ for S over A/J, where $Q_s/\tau_P(Q_s) = 0$ if $Q_s = P$, so $\operatorname{pd}_{A/J} S \geq \operatorname{pd}_A S - 1$. Since $\operatorname{pd}_A P/\underline{\mathbf{r}}P = 0$, we have $\operatorname{gldim} A \leq \operatorname{gldim} A/J + 1$. □

We now prove the main result.

Proposition 2.3. *Let A be a serial ring with $P_1, \ldots, P_n$ a complete set of non-isomorphic indecomposable projective A-modules.*

a) If there is a simple projective A-module then

$$\text{gldim}\, A \leq \lambda(A) \leq n-1$$

b) If there is no simple projective A-module and A is quasihereditary then

$$\text{gldim}\, A \leq \lambda(A) + 1 \leq n$$

Proof: The proof is by induction on $\lambda(A)$. Assume $\lambda(A) = 1$. If A has a simple projective, then $\text{gldim}\, A = 1 = \lambda(A)$, by Lemma 1.1. If A has no simple projective module, then $\text{gldim}\, A = 2 = \lambda(A) + 1$, again by Lemma 1.1, since A must be of finite global dimension, being quasihereditary.

Now assume that $\lambda(A) > 1$. We first consider the case when A has a simple projective P. Let $J = \tau_P(A)$. By Lemma 1.4, $\lambda(A) - 1 \leq \lambda(A/J) \leq \lambda(A)$. If $\lambda(A/J) = \lambda(A) - 1$, then

$$\begin{aligned} \text{gldim}\, A &\leq \text{gldim}\, A/J + 1, \text{by Lemma 2.2} \\ &\leq \lambda(A/J) + 1, \text{by induction} \\ &= (\lambda(A) - 1) + 1 \\ &= \lambda(A) \end{aligned}$$

If $\lambda(A) = \lambda(A/J)$ then by Lemma 2.1, every simple $S \ncong P$ has $\text{pd}_A S = \text{pd}_{A/J} S$. Since $\text{pd}_A P = 0$, we have $\text{gldim}\, A = \text{gldim}\, A/J$ and $\lambda(A/J) \leq n-2$. A/J contains a simple projective and $\lambda(A/J) \leq n-2$, so the induction continues by factoring out an heredity ideal generated by a simple projective until λ decreases. Since λ is bounded by the number of simple modules, which decreases by one after factoring out the heredity ideal, λ must decrease eventually. Therefore (a) holds.

Now assume that A is quasihereditary and has no simple projective. Then there is an ℓ-projective P with $\underline{\mathbf{r}}P$ not projective. Let $J = \tau_P(A)$. If $\lambda(A/J) = \lambda(A) - 1$, then A/J has a simple projective so we may use (a) to get that

$$\begin{aligned} \text{gldim}\, A &\leq \text{gldim}\, A/J + 2, \text{by } [\mathbf{DR}] \\ &\leq \lambda(A/J) + 2 \\ &= (\lambda(A) - 1) + 2 \\ &= \lambda(A) + 1 \end{aligned}$$

If $\lambda(A/J) = \lambda(A)$ then by Lemma 2.1 we have $\text{pd}_A S = \text{pd}_{A/J} S$ for every $S \not\cong P/\underline{\mathbf{r}}P$. The projective dimension of a module is the projective dimension of its composition factor of maximal projective dimension. Since $\underline{\mathbf{r}}P$ has no composition factor isomorphic to $P/\underline{\mathbf{r}}P$, we have

$$\begin{aligned}\text{pd}_A \underline{\mathbf{r}}P &= \max\{\text{pd}_A S \mid S \text{ is a composition factor of } \underline{\mathbf{r}}P\} \\ &\leq \max\{\text{pd}_A S \mid S \not\cong P/\underline{\mathbf{r}}P\} \\ &= \max\{\text{pd}_{A/J} S \mid S \not\cong P/\underline{\mathbf{r}}P\} \\ &= \text{gldim}\, A/J\end{aligned}$$

Therefore $\text{pd}_A P/\underline{\mathbf{r}}P \leq \text{gldim}\, A/J + 1$, which combined with (a) implies that

$$\begin{aligned}\text{gldim}\, A &\leq \text{gldim}\, A/J + 1 \\ &\leq \lambda(A/J) + 1 \\ &= \lambda(A) + 1,\end{aligned}$$

which completes the proof. □

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