A5

Summary of Set theory

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1. Informal Set Theory

1. Introduction

Metalogic is formulated in a language which is basically English [supposing that is the language of use] enhanced by numerous set-theoretic concepts. For that reason, in the current chapter, we examine some of the most basic concepts of set theory.

2. Sets

Fundamental to set theory is the notion of *membership*; sets have members, also called elements. We write '$a \in A$' to say that a is a member (element) of A, and we write '$a \notin A$' to say that a is not a member of A. Sets are totally identified by their membership. If A and B are sets, and are distinct (i.e., $A \neq B$), then there must be an element of A which is not an element of B, or vice versa. This is succinctly stated in the principle of *extensionality*:

Ext

$$\forall x(x \in A \leftrightarrow x \in B) \rightarrow A=B$$

Here, the tacit domain is the domain of sets.

Sets can have various things as elements, including other sets. A set whose elements are all sets is called a *pure set*; a set that has at least one element which is not a set is called an *impure set*. Formal set theory is concerned exclusively with pure sets; informal set theory is concerned primarily with impure sets. Informal set theory begins with an existing domain of objects, presumed not to be sets, and constructs all sets over that domain. For example, one can construct sets over the domain of natural numbers, or over the domain of persons, or whatever. By contrast, pure set theory assumes no pre-existing domain, but builds all sets out of "thin air". As it is ordinarily understood, metalogic operates in the realm of informal (impure) set theory; in particular, linguistic symbols are regarded as non-sets, as are many of the objects designated by these symbols (e.g., truth values).

3. Convenient Notation for Simple Cases

We now make explicit an informal convention, already employed above — to use lower case Roman letters to denote "points", upper case Roman letters to denote sets whose elements are points, and upper case script letters to denote sets whose elements are sets. Some sort of convention like this is occasionally useful in visually clarifying the hierarchy of sets, and we will use such a convention when it is helpful. For example, we might write

$$a \in B \ \& \ B \in \mathcal{C}$$

even though the following is equally legitimate.

$$a \in b \ \& \ b \in c.$$

Given how complex sets (of sets… of sets…) are, no simple-minded syntactic convention can possibly do justice to the richness of possible relations among sets. Still it is occasionally useful to use such a convention for the simple sorts of sets.

4. Set Abstracts

The customary way to denote a set with finitely many elements is to list the elements, then surround the list with curly brackets. The following are examples.

$\{1\}, \{1,2\}, \{1,2,3,\}, \{2,3\}, \{2,3,4\}$, etc.

The basic principle about such expressions is formalized as follows.

$\forall x(x \in \{a\} \leftrightarrow x=a)$

$\forall x(x \in \{a,b\} \leftrightarrow. \ x=a \vee x=b)$

$\forall x(x \in \{a,b,c\} \leftrightarrow. \ x=a \vee x=b \vee x=c)$

etc.

Terminology: {a} is the *singleton* of a; {a,b} is the *doubleton* of a and b; etc.

More generally, we can form *set abstracts* of the form $\{\nu{:}\mathbb{F}\}$, where ν is a variable and $\mathbb{F}$ is a formula. The following are informal examples.

{x: x is an even}
{x: x is a Republican}
{x: x is less than 3 and x is greater than 2}

An object of the domain is an element of the set $\{\nu{:}\mathbb{F}\}$ if and only if that object satisfies the formula $\mathbb{F}$. Generally, the set denoted by a given abstract will depend on the domain of discourse (natural numbers, real numbers, persons, etc.)

Notice that all the simple set-denoting expressions above can be defined using this more general notation, as follows.

(d0.1)	$\{a\}$	$=_{df}$	$\{x: x=a\}$	[singleton]
(d0.2)	$\{a,b\}$	$=_{df}$	$\{x: x=a \vee x=b\}$	[doubleton]
	etc.			

5. Set-Theoretic Operations

Various set-theoretic operations can also be defined using set abstracts. The following are a few examples.

(d1)	$A\cap B$	$=_{df}$	$\{x: x\in A \ \& \ x\in B\}$	[simple intersection]
(d2)	$A-B$	$=_{df}$	$\{x: x\in A \ \& \ x\neq B\}$	[set-difference]
(d3)	$A\cup B$	$=_{df}$	$\{x: x\in A \lor x\in B\}$	[simple union]
(d4)	$\bigcup(\mathcal{C})$	$=_{df}$	$\{x: \exists Y(Y\in \mathcal{C} \ \& \ x\in Y)\}$	[general union]
(d5)	$\bigcap(\mathcal{C})$	$=_{df}$	$\{x: \forall Y(Y\in \mathcal{C} \to x\in Y)\}$	[general intersection]

Whereas simple intersection/union applies to pairs of sets (and to finite collections by inductive generalization), general intersection/union applies to arbitrary collections of sets, including infinitely large collections. Simple intersection/union is a special case of general intersection/union, just as conjunction/disjunction is a special is a special case of universal/existential quantification. In particular, we have the following.

(t1) $\bigcap\{A,B\} = A\cap B$

(t2) $\bigcup\{A,B\} = A\cup B$

6. The Empty Set

Generally, sets have elements. There is an exception; set theory assumes the existence of a set with no elements — the *empty set*, denoted $\varnothing$. Extensionality entails that every set with no elements is identical to $\varnothing$, the word 'the' in '*the* empty set' is logically justified.

7. Inclusion, Exclusion, Subsets, and Supersets

Set A is said to be a *subset* of set B iff every element of A is an element of B. Alternative terminology: A is *included* in B.

(d6)	$A \subseteq B =_{df} \forall x(x \in A \rightarrow x \in B)$

The formal definition presupposes A and B are sets. Examples: the set of sophomores is a subset of the set of students, which is in turn a subset of the set of humans. Notice that the empty set is a subset of every set. Notice also that every set is a subset of itself. A is a *proper subset* of B iff A is a subset of B, and A is distinct of B.

(d7)	$A \subset B =_{df} A \subseteq B \ \& \ A \neq B$

Two sets A,B are said to be *disjoint* iff no element of A is an element of B; formally:

(d8)	$A \perp B =_{df} {\sim}\exists x(x \in A \ \& \ x \in B\}$

Given any set A, we can form the *power set* of A, which contains precisely the subsets of A.

(d9)	$\wp(A) =_{df} \{X: X \subseteq A\}$

Examples:

If $A = \varnothing$, then $\wp(A) = \{\varnothing\}$
If $A = \{1\}$, then $\wp(A) = \{ \varnothing, \{1\} \}$
If $A = \{1,2\}$, then $\wp(A) = \{ \varnothing, \{1\}, \{2\}, \{1,2\} \}$
If $A = \{1,2,3\}$, then $\wp(A) = \{ \varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

The power set $\wp(A)$ is bigger than A; indeed, if A has n elements, then $\wp(A)$ has 2^n elements.

8. Ordered Pairs, Ordered Triples, etc.

The elements of a set are not ordered, even though their names may be. The following follows from the principle of extensionality.

(1) $\{a,b\} = \{b,a\}$
(2) $\{a,b,c\} = \{a,c,b\} = \{b,a,c\} = \{b,c,a\} = \{c,a,b\} = \{c,b,a\}$
etc.

In addition to unordered collections (doubletons, tripletons, etc.), there are also *ordered pairs*, *ordered triples*, etc. These are also called 2-tuples, 3-tuples, etc. It is customary to use round parentheses to denote such collections: e.g., (a,b), (a,b,c), (a,b,c,d), etc. In particular, (a,b) is the ordered pair whose first term (component) is a and whose second term (component) is b.

Just as the principle of extensionality governs the individuation of sets, the following fundamental principles govern the individuation of n-tuples.

(p.1)	(a,b) = (p,q)	$\leftrightarrow$.	a=p & b=q
(p.2)	(a,b,c) = (p,q,r)	$\leftrightarrow$.	a=p & b=q & c=r
	etc.		

The above principles give formal criteria of individuation, they do not materially identify ordered pairs. Similarly, the principle of extensionality offers the formal criteria of individuation for sets in general, but it does not materially identity them.

9. The Cartesian Product

Given any pair of sets A,B, one can form the *Cartesian product* of A with B, which is denoted $A\times B$. Something is an element of $A\times B$ iff it is an ordered pair whose first term is in A and whose second term is in B. In other words

(p) $p \in A\times B \leftrightarrow$ p is an ordered pair & $1st(p) \in A$ & $2nd(p) \in B$

If p is (a,b), then 1st(p) = a, and 2nd(p) = b. The Cartesian product can also be defined as follows.

(d9) $A\times B =_{df} \{(x,y): x \in A \ \& \ y \in B\}$

The latter singular term is a *generalized set abstract* of the form $\{\tau:\mathbb{F}\}$, where τ is a singular term, and $\mathbb{F}$ is a formula. Simple examples from English will suffice to explain the basic idea.

(1) {the mother of x: x is a Democrat}
(2) $\{x^2$: x is even$\}$
(3) {x+y: x,y less than 10}

Set #1 is the set of mothers of Democrats; set #2 is the set of squares of even numbers; set #3 is the set of numbers that result from adding two numbers less than 10.

The product defined above is the binary Cartesian product. In general, for any number n, there is the n-fold Cartesian product. The following are the definitions.

(d10.1)	$\times(A,B,C)$	$=_{df} \{(a,b,c): a\in A \ \& \ b\in B \ \& \ c\in C\}$
(d10.2)	$\times(A,B,C,D)$	$=_{df} \{(a,b,c,d): a\in A \ \& \ b\in B \ \& \ c\in C \ \& \ d\in D\}$
	etc.	

10. Relations

Set-theoretic relations are intended to be the extensions of polyadic predicates, in a way that ordinary sets are extensions of monadic predicates. For example, the extension of the predicate '...is taller than...' is the set of ordered pairs whose first term is taller than the second term.

Abstracting from the syntactic motivation, we define a binary (2-place) relation simply to be *any* set of ordered pairs; more generally, we define an n-place relation to be any set of n-tuples.

When used by itself, the term 'relation' is intended to mean 'binary relation'. Let us concentrate on these for the moment.

(D1) R is a relation $=_{df}$ R is a set of ordered pairs

Of course, the latter means that every element of R is an ordered pair.

Notation: we say that element a bears relation R to element b iff the ordered pair (a,b) is an element of R:

(d11) a bears R to b $=_{df}$ $(a,b) \in R$

It is customary to abbreviate the 'bears' predicate in the starkest manner possible:

(d12) aRb $=_{df}$ a bears R to b $=_{df}$ $(a,b) \in R$

Associated with every relation R are three inter-related sets, called the *domain*, *range*, and *field*, of R. These are officially defined as follows.

(d13) dom(R) $=_{df}$ $\{1st(p): p \in R\} = \{x: \exists y[xRy]\}$

(d14) ran(R) $=_{df}$ $\{2nd(p): p \in R\} = \{y: \exists x[xRy]\}$

(d15) fld(R) $=_{df}$ $dom(R) \cup ran(R)$

(D2) R is a relation *from* A *to* B iff it satisfies the following restrictions

(1) $dom(R) \subseteq A$
(2) $ran(R) \subseteq B$

(D3) R is a relation *on* A iff $fld(R) \subseteq A$.

Notice that (1) and (2) simply amount to: $R \subseteq A \times B$. Also, notice that a relation on A is a simply a relation from A to A.

11. Functions

From the point of view of set theory, (unary) functions are a species of (binary) relation. In addition to unary (1-place) functions, there are also 2-place, 3-place, etc., functions. When used by itself 'function' means 'unary (1-place) function'. The following is the official definition.

(D4) A function is, by definition, a relation R satisfying the following restriction.
(r) $\forall$xyz(xRy & xRz .$\rightarrow$ y=z)

Notice that restriction is equivalent to:

(r*) ~$\exists$xyz(xRy & xRz & y$\neq$z)

In other words, a function is a relation R in which no single thing bears R to more than one thing. It is customary (but not universal) to use lower case letter to denote functions. Also, in connection with functions, there is additional notation, namely, function-argument-value notation, formally introduced as follows.

(d16) f(a) $=_{df}$ ιx[aRx]

Suppose f is a function, and suppose a $\in$ dom(f). Then a bears f to exactly one thing, which is denoted f(a). So, in order to convey the fact that a bears f to b, we say that b=f(a). In other words, assuming f is a function, and a is in the domain of f, we have

(t) b=f(a) $\leftrightarrow$ afb

Terminology: it is customary to think of functions as "taking" input and "producing" output; whereas the input, a, is called the *argument*, the output, f(a), is called the *value* of the function at the argument a.

Examples:

The set of ordered pairs satisfying:

$x^2 + y^2 = 4$ is *not* a function.

On the other hand, the set of ordered pairs satisfying:

$y = x^2$ is a function.

The set of ordered pairs satisfying:

y parents x is *not* a function.

On the other hand, the set of ordered pairs satisfying:

y fathers x is a function.

12. Functions — Into, Onto, 1-1

Just as with relations, one can define when a function is from one set to another; there is a slight difference in restriction (1).

> (D5) Where A,B are sets, a function *from* A *into* B is, by definition a function satisfying the following restrictions:
>
> (1) dom(f) = A
> (2) ran(f) $\subseteq$ B
>
> Notation: f:A→B $=_{df}$ f is a function from A into B.

Similarly, one can define a function to be from A *onto* B, as follows.

> (D6) Where A,B are sets, a function *from* A *onto* B is, by definition a function satisfying the following restrictions:
>
> (1) dom(f) = A
> (2) ran(f) = B
>
> Notation: f:A→B(onto)

Functions, in general, can assign the same output to a given input; two people can have the same father; two numbers can have the same square. Functions that assign different output to different input are of special interest, and are called 1-1 (one-one; one-to-one) functions. These are officially defined as follows.

> (D7) A *1-1 function* is a function satisfying the following restriction.
>
> (r) $\forall x \forall y \{f(x)=f(y) \rightarrow x=y\}$
>
> [Here, the variables are understood to range over the domain of f.]
>
> Notation: f:A→B(1-1)

A function that is both 1-1 and onto is called a *bijection*, or a one-to-one correspondence. More about these later.

13. Multi-Place Functions and Relations

So far, we have defined unary functions in terms of binary relations. We still need to define 2-place, 3-place, etc., functions. There are a couple of alternatives. On the one hand, we can define an n-place function to be a special kind of n+1-place relation. On the other hand, we can define an n-place function to be a special kind of unary function.

We choose the latter approach. First a few subordinate definitions.

(D8)	An n-tuple of elements of A is, by definition, an n-tuple every term of which is an element of A.

(D9)	The n-fold Cartesian power of a set A, denoted A^n, is the set of n-tuples of elements of A

Instances:

$A^2 = \{(a,b): a,b \in A\}$
$A^3 = \{(a,b,c): a,b,c \in A\}$
etc.

Notice that $A^2 = A \times A$, $A^3 = \times(A,A,A)$, $A^4 = \times(A,A,A,A)$.

We can now define n-place relation on A, and n-place function from A to B.

(D10)	R is an n-place relation on A iff $R \subseteq A^n$.
(D11)	f is an n-place function from A iff f is a (unary) function from A^n into B.
(D12)	f is an n-place function on A iff f is a (unary) function from A^n into A.

14. Cardinality

The cardinality of a set is how big it is. The concept of size of sets ultimately rests on the concept of sameness of size. This is not circular, since sameness of size can be defined independently of size. First, the definition of bijection.

(D13)	A *bijection* between A and B is, by definition, any 1-1 function from A onto B. [Notation: $f:A \leftrightarrow B$]

Given the notion of bijection, we can define sameness of size, which is called *equipollence*, and denoted $\approx$.

(D14) $A \approx B =_{df} \exists f$: f is a bijection between A and B

When there is a bijection between two sets A,B, they can be paired in a one-to-one manner; each element of A is matched to an element of B.

In the case of finite sets, equipollence is intuitively clear. If A has n elements, then A is equipollent to any other set with n elements. Infinite sets are less obvious. The set $\mathcal{N}$ of natural numbers is an infinite set. A set is said to be *denumerable* iff it is equipollent to the set $\mathcal{N}$; A is *countable* iff it is either finite or denumerable. A countably infinite set is a denumerable set. Every infinite subset of N is denumerable; for example, the set of even numbers, the set of odd numbers, the set of prime numbers. These are proper subsets of $\mathcal{N}$, but they are also equipollent to $\mathcal{N}$! Other denumerable sets include the set of integers (positive and negative), and the set of rational numbers.

Many infinite sets are denumerable, but not every infinite set is denumerable. The power set $\mathcal{P}(\mathcal{N})$ is *uncountable* (i.e., infinite, but not denumerable), so is the set of irrational numbers, and the set of real numbers.

How many different infinite sizes are there? A bunch! Cantor first proved that, no matter how big a set is, its power set is bigger. So, $\mathcal{N}$ is smaller than $\mathcal{P}(\mathcal{N})$, which is smaller than $\mathcal{P}\mathcal{P}(\mathcal{N})$, which is smaller than $\mathcal{P}\mathcal{P}\mathcal{P}(\mathcal{N})$, and so forth.

Summary of Definition of Sizes

D15	S is **finite** $=_{df}$ S is equipollent to the set of predecessors of some natural number
D16	S is **infinite** $=_{df}$ S is not finite
D17	S is **denumerably-infinite** $=_{df}$ S is equipollent to the set $\mathcal{N}$ of natural numbers
D18	S is **uncountable** (non-denumerable) $=_{df}$ S is neither finite nor denumerably infinite

Note carefully that some authors use the term ‘denumerable’ to mean ‘denumerably-infinite’; other authors use the term ‘denumerable’ to mean ‘finite *or* denumerably-infinite’, in which case means the same as ‘countable’.

2. Formal Set Theory

1. The Formal Language of Set Theory

Pure set theory can be formulated in a first order language, specified as follows.

1. Vocabulary

1. Logical Vocabulary

the usual symbols/characters, but expanded to include letters of various fonts for variables and constants.

2. Non-Logical Vocabulary

$\in$: a 2-place predicate, written in infix notation.

'$[\alpha\in\beta]$' is read 'α is an element of β'.

As usual, the outer brackets are dropped in many circumstances.

2. Rules of Formation

1. Singular Terms

every variable is a singular term;
every constant is a singular term;
there are no (primitive) proper nouns;
there are no (primitive) function signs;
if $\mathbb{F}$ is a formula, and ν is a variable, then $\iota\nu\mathbb{F}$ is a singular term;
nothing else is a singular term.

2. Atomic Formulas

if τ_1 and τ_2 are singular terms, then $[\tau_1\in\tau_2]$ is an atomic formula;
if τ_1 and τ_2 are singular terms, then $[\tau_1=\tau_2]$ is an atomic formula;
[Note: brackets are optional; sometimes they help; sometimes they hinder.]

3. Formulas

every atomic formula is a formula;
if $\mathbb{F}$ is a formula, then so is: $\sim\mathbb{F}$;
if $\mathbb{F}$ and $\mathbb{G}$ are formulas, then so are: $(\mathbb{F}\rightarrow\mathbb{G})$, $(\mathbb{F}\vee\mathbb{G})$, $(\mathbb{F}\&\mathbb{G})$, $(\mathbb{F}\leftrightarrow\mathbb{G})$;
if $\mathbb{F}$ is a formula, and ν is a variable, then $\forall\nu\mathbb{F}$ and $\exists\nu\mathbb{F}$ are formulas;
nothing else is a formula.

2. The Axioms of Set Theory

The following are the principal axioms of pure set theory (in the tradition of Zermelo and Fraenkel; what is accordingly called ZF set theory). Because this is *pure* set theory, the universe of discourse (domain) consists exclusively of sets, and so the quantifiers range over sets.

A1.	$\forall x \forall y[\forall z(z\in x \leftrightarrow z\in y) \rightarrow x=y]$	[Extensionality]
A2.	$\exists x{\sim}\exists y[y\in x]$	[Empty Set]
*A3.	$\forall x\exists y\forall z[z\in y \leftrightarrow (z\in x \;\&\; \mathbb{F}]$	[Separation]
A4	$\forall x\forall y\exists z\forall w(w\in z \leftrightarrow. w=x \vee w=y)$	[Pairs]
A5.	$\forall x\exists y\forall z[z\in y \leftrightarrow \exists w(w\in x \;\&\; z\in w)]$	[Unions]
A6.	$\forall x\exists y\forall z[z\in y \leftrightarrow \forall w(w\in z \rightarrow w\in x)]$	[Power Sets]
*A7.	$\forall \nu\exists!\upsilon\mathbb{F} \rightarrow \forall x\exists y\forall z\{z\in y \leftrightarrow \exists w(w\in x \;\&\; \mathbb{F}[w/\nu,z/\upsilon])\}$	[Replacement]

*An axiom schema; $\mathbb{F}$ is any formula in which y is not free.

Note: '∃!' is the exactly-one quantifier, which may be defined as follows.

(d) $\exists!\upsilon\mathbb{F} \;=_{df}\; \exists\upsilon(\mathbb{F} \;\&\; \forall\omega(\mathbb{F}[\omega/\upsilon] \leftrightarrow \omega=\upsilon))$

Here, $\mathbb{F}$ is any formula in which variable υ is free for variable ω.

3. Definitions of Non-Primitive Notation

Metalinguistic Convention: For the sake of visual sanity we borrow constants ['a', 'b', 'A', 'B', '$\mathcal{A}$', '$\mathcal{B}$', etc.] and variables ['x', 'y', 'z', etc.] from the object language to use as meta-linguistic variables. Whereas we use "constants" to schematically represent *closed singular terms* of the object language, we use "variables" to schematically represent variables in the object language.

1.	$a \neq b$	$=_{df}$	$\sim[a=b]$	[negation]
2.	$a \notin b$	$=_{df}$	$\sim[a \in b]$	[negation]
3.	$A \subseteq B$	$=_{df}$	$\forall x(x \in A \rightarrow x \in B)$	[inclusion]
4.	$A \subset B$	$=_{df}$	$A \subseteq B \;\&\; A \neq B$	[proper inclusion]
5.	$A \supseteq B$	$=_{df}$	$B \subseteq A$	[converse inclusion]
6.	$A \supset B$	$=_{df}$	$B \subset A$	[converse proper inclusion]
7.	$A \perp B$	$=_{df}$	$\sim\exists x(x \in A \;\&\; x \in B)$	[exclusion]
8.	$\{\nu: \mathbb{F}\}$	$=_{df}$	$\iota S \forall \nu(\nu \in S \leftrightarrow \mathbb{F})$ [S not free in $\mathbb{F}$]	[set-abstract]
9.	$\{a\}$	$=_{df}$	$\{x: x=a\}$	[singleton]
	$\{a,b\}$	$=_{df}$	$\{x: x=a \vee x=b\}$	[doubleton]
	$\{a,b,c\}$	$=_{df}$	$\{x: x=a \vee x=b \vee x=c\}$	[tripleton]
	etc.			
10.	$\mathbb{U}$	$=_{df}$	$\{x: x=x\}$	[universal set]
11.	$\varnothing$	$=_{df}$	$\{x: x \neq x\}$	[empty set]
12.	$A \cap B$	$=_{df}$	$\{x: x \in A \;\&\; x \in B\}$	[simple intersection]
13.	$A - B$	$=_{df}$	$\{x: x \in A \;\&\; x \notin B\}$	[set-difference]
14.	$A \cup B$	$=_{df}$	$\{x: x \in A \vee x \in B\}$	[simple union]
15.	$A + B$	$=_{df}$	$(A-B) \cup (B-A)$	[Boolean sum]
16.	$\bigcup(\mathcal{C})$	$=_{df}$	$\{x: \exists y(y \in \mathcal{C} \;\&\; x \in y)\}$	[general union]
17.	$\bigcap(\mathcal{C})$	$=_{df}$	$\{x: \forall y(y \in \mathcal{C} \rightarrow x \in y)\}$	[general intersection]
18.	$\wp(A)$	$=_{df}$	$\{X: X \subseteq A\}$	[power set]
19.	$\exists!\nu\mathbb{F}$	$=_{df}$	$\exists \upsilon \forall \nu(\mathbb{F} \leftrightarrow \nu=\upsilon)$ [υ not free in $\mathbb{F}$]	[unique existence]
20.	$\Sigma\nu\mathbb{F}$	$=_{df}$	$\exists! S \forall \nu(\nu \in S \leftrightarrow \mathbb{F})$ [S not free in $\mathbb{F}$]	[legitimacy]
21.	$\{\tau: \mathbb{F}\}$	$=_{df}$	$\{\nu: \exists x(\mathbb{F} \;\&\; \nu=\tau(x))\}$	[general abstract]
	[here, τ is any singular term, $\mathbb{F}$ is any formula; ν is any variable not free in τ or $\mathbb{F}$; x are all the free variables common to τ and $\mathbb{F}$]			
22.	$\forall xy$	$=_{df}$	$\forall x \forall y$	
	$\exists xy$	$=_{df}$	$\exists x \exists y$	[quantifier abbr.]
23.	$a,b \in S$	$=_{df}$	$a \in S \;\&\; b \in S$	
	$a \in S,T$	$=_{df}$	$a \in S \;\&\; a \in T$	[&-$\in$ abbr.]

4. Examples of Theorems

Note: Unless otherwise specified, every theorem is to be understood as universally quantified over all variables logically permitted to generalize the constants ‘a’, ‘A’, ‘b’, ‘B’, ‘C’, etc. Alternatively, we can treat the constants as meta-linguistic variables ranging over closed singular terms (as in the definitions), in which case each theorem is a theorem *schema*.

1. $A=B \rightarrow \forall x(x\in A \leftrightarrow x\in B)$
2. $A\neq B \leftrightarrow \exists x([x\in A \;\&\; x\notin B] \vee [x\in B \;\&\; x\notin A])$
3. $A \subseteq A$
4. $A \subseteq B \;\&\; B \subseteq A \;.\!\!\rightarrow A=B$
5. $A \subseteq B \;\&\; B \subseteq C \;.\!\!\rightarrow A \subseteq C$
6. $\sim[A \subset A]$
7. $A \subset B \rightarrow \sim[B \subset A]$
8. $A \subset B \;\&\; B \subset C \;.\!\!\rightarrow A \subset C$
9. $A \subset B \;\rightarrow\; \sim[B \subseteq A]$
10. $A \subset B \leftrightarrow.\; A \subseteq B \;\&\; \sim[B \subseteq A]$
11. $A \subseteq B \;\&\; B \perp C \;.\!\!\rightarrow\; A \perp C$
12. $A \perp B \;\leftrightarrow\; \sim\exists x[x\in A \;\&\; x\in B]$
13. $A \perp A \;\leftrightarrow\; \sim\exists x[x\in A]$
14. $\sim\exists x[x\in A] \rightarrow A \subseteq B$
15. $\sim\exists x[x\in A] \rightarrow A \perp B$
16. $\sim\exists x[x\in \varnothing]$
17. $\varnothing \subseteq A$
18. $\varnothing \perp A$
19. $\sim\exists x[x\in A] \leftrightarrow A=\varnothing$
20. $A \subseteq \varnothing \;\rightarrow\; A=\varnothing$
21. $A\cap A = A$
22. $A\cap B = B\cap A$
23. $A\cap(B\cap C) = (A\cap B)\cap C$
24. $A\cap B = A \;\leftrightarrow\; A \subseteq B$
25. $A\cap\varnothing = \varnothing$
26. $A \subseteq B \;\&\; A \subseteq C \;.\!\!\rightarrow\; A \subseteq B\cap C$
27. $A \perp B \;\leftrightarrow\; A\cap B = \varnothing$
28. $A\cup A = A$
29. $A\cup B = B\cup A$
30. $A\cup(B\cup C) = (A\cup B)\cup C$
31. $A\cup B = B \;\leftrightarrow\; A \subseteq B$
32. $A\cup\varnothing = A$
33. $A \subseteq C \;\&\; B \subseteq C \;.\!\!\rightarrow\; A\cup B \subseteq C$
34. $A\cap(A\cup B) = A$
35. $A\cup(A\cap B) = A$
36. $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$
37. $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$
38. $(A\cap B)\cup(A\cap C)\cup(B\cap C) = (A\cup B)\cap(A\cup C)\cap(B\cup C)$
39. $A-A = \varnothing$
40. $A-(A-B) = A\cap B$

41. $A-(A\cap B) = A-B$
42. $(A-B)-C = (A-C)-B$
43. $A-(B-A) = A$
44. $A-B \perp B$
45. $A \subseteq B \;\leftrightarrow\; A-B = \varnothing$
46. $A \perp B \;\leftrightarrow\; A-B = A$
47. $A-(B\cap C) = (A-B)\cup(A-C)$
48. $A-(B\cup C) = (A-B)\cap(A-C)$
49. $A+A = \varnothing$
50. $A+B = B+A$
51. $A+(B+C) = (A+B)+C$
52. $A+B = \varnothing \;\rightarrow\; A \subseteq B$
53. $A+B = \varnothing \;\leftrightarrow\; A=B$
54. $A\cup B = (A+B)+(A\cap B)$
55. $A\cap B = (A+B)+(A\cup B)$
56. $A+B = (A\cup B)+(A\cap B)$
57. $(A\cap B)\cap C \subseteq (A+B)+C$
58. $A+B = A\cup B \;\leftrightarrow\; A \perp B$
59. $A\cap(A+B) \subseteq A-B$
60. $a\in\{a\}$
61. $a\in\{a,b\}$ & $b\in\{a,b\}$
62. $\{a,b\}=\{c\} \rightarrow a=b$
63. $a=b \rightarrow \{a,b\}=\{a\}$
64. $\{a\}\cup\{b\} \subseteq \{c\} \;\rightarrow\; a=b$
65. $\{a,b\} = \{c,d\} \;\leftrightarrow.\; (a=c \,\&\, b=d) \vee (a=d \,\&\, b=c)$
66. $\bigcup\{A\} = A$
67. $\bigcap\{A\} = A$
68. $\bigcup\{A,B\} = A\cup B$
69. $\bigcap\{A,B\} = A\cap B$
70. $\bigcup(\varnothing) = \varnothing$
71. $\bigcup(\{\varnothing\}) = \varnothing$
72. $\sim\exists x[x = \bigcap(\varnothing)]$
73. $\bigcap(\{\varnothing\}) = \varnothing$
74. $\bigcup(A\cup B) = \bigcup(A) \cup \bigcup(B)$
75. $\bigcap(A\cup B) = \bigcap(A) \cap \bigcap(B)$
76. $\forall X(X\in \mathcal{C} \rightarrow X \subseteq B) \rightarrow \bigcup(\mathcal{C}) \subseteq B$
77. $\forall X(X\in \mathcal{C} \rightarrow X \perp B) \rightarrow \bigcup(\mathcal{C}) \perp B$
78. $\forall X(X\in \mathcal{C} \rightarrow B \subseteq X) \rightarrow B \subseteq \bigcap(\mathcal{C})$
79. $\varnothing \in \Xi(A)$
80. $A \in \Xi(A)$
81. $A \subseteq B \;\leftrightarrow\; \Xi(A) \subseteq \Xi(B)$
82. $\Xi(A\cap B) = \Xi(A) \cap \Xi(B)$
83. $\Xi(A) \cup \Xi(B) \subseteq \Xi(A\cup B)$
84. $\bigcup(\Xi(A)) = A$
85. $\bigcap(\Xi(A)) = \varnothing$
86. $A\in \mathcal{C} \rightarrow A \subseteq \bigcup(\mathcal{C})$

87. $A \in C \rightarrow \bigcap(C) \subseteq A$

3. A Formal Theory of Functions

1. Introduction

In this appendix, we examine a formal theory of functions, a theory that does not reduce functions to sets, but treats them as further primitive objects, on a par with sets. Note that the theory does not say that functions are not sets; rather, it is non-committal.

2. Primitive Concepts (Vocabulary)

Set	:	one-place predicate
Fun	:	one-place predicate
$\in$	:	two-place predicate
app	:	two-place function sign
$\varnothing$	:	proper noun

readings:

Set[α]	:	α is a set
Fun[α]	:	α is a function
$\alpha \in \beta$	:	α is an element of β [α is a member of β]
app(α,β)	:	the result of applying α to β [what α assigns to β]
$\varnothing$	:	the empty set

3. Notational Shorthand

In the following, the lower case Greek letters are metalinguistic variables standing for arbitrary closed singular terms. The restriction to closed singular terms means, in particular, that these definitions can be applied inside derivations only after free variables are replaced by constants. Whenever the variable ‘x’ is used, it is understood that it is not free in $\mathbb{F}$.

(1)	$\alpha \notin \beta$	$=_{df}$	$\sim[\alpha \in \beta]$	
(2)	$\alpha,\beta \in \gamma$	$=_{df}$	$\alpha\in\gamma \;\&\; \beta\in\gamma$	
(3)	$\alpha \in \beta,\gamma$	$=_{df}$	$\alpha\in\beta \;\&\; \alpha\in\gamma$	
(4)	$\forall\nu_1\nu_2\mathbb{F}$	$=_{df}$	$\forall\nu_1\forall\nu_2\mathbb{F}$	
(5)	$\exists\nu_1\nu_2\mathbb{F}$	$=_{df}$	$\exists\nu_1\exists\nu_2\mathbb{F}$	
(6)	$\forall\mathbb{F}\mathbb{G}$	$=_{df}$	$\forall\nu\{\mathbb{F} \rightarrow \mathbb{G}\}$	[ν is the unique variable free in $\mathbb{F}$ and $\mathbb{G}$]
(7)	$\exists\mathbb{F}\mathbb{G}$	$=_{df}$	$\exists\nu\{\mathbb{F} \;\&\; \mathbb{G}\}$	[ν is the unique variable free in $\mathbb{F}$ and $\mathbb{G}$]
(8)	$\iota\mathbb{F}\mathbb{G}$	$=_{df}$	$\iota\nu\{\mathbb{F} \;\&\; \mathbb{G}\}$	[ν is the unique variable free in $\mathbb{F}$ and $\mathbb{G}$]
(9)	$\forall S\nu\mathbb{F}$	$=_{df}$	$\forall\nu\{Set[\nu] \rightarrow \mathbb{F}\}$	
(10)	$\exists S\nu\mathbb{F}$	$=_{df}$	$\exists\nu\{Set[\nu] \;\&\; \mathbb{F}\}$	
(11)	$\iota S\nu\mathbb{F}$	$=_{df}$	$\iota\nu\{Set[\nu] \;\&\; \mathbb{F}\}$	
(12)	$\forall s_m\mathbb{F}$	$=_{df}$	$\forall Sx_m\mathbb{F}[x_m/s_m]$	
(13)	$\exists s_m\mathbb{F}$	$=_{df}$	$\exists Sx_m\mathbb{F}[x_m/s_m]$	
(14)	$\iota s_m\mathbb{F}$	$=_{df}$	$\iota Sx_m\mathbb{F}[x_m/s_m]$	
(15)	repeat the above for ‘Fun’, ‘F’, ‘f’.			
(16)	$\exists!\nu\mathbb{F}$	$=_{df}$	$\exists x\forall\nu\{\mathbb{F} \leftrightarrow \nu=x\}$	
(17)	$E![\alpha]$	$=_{df}$	$\exists x[x = \alpha]$	

4. Derivative Concepts (Vocabulary)

(1)	Uhr[α]	$=_{df}$	E![α] & ~Set[α] & ~Fun[α]
(2)	$\alpha \subseteq \beta$	$=_{df}$	Set[α] & Set[β] & $\forall x\{x \in \alpha \rightarrow x \in \beta\}$
(3)	$\alpha \subset \beta$	$=_{df}$	$\alpha \subseteq \beta$ & $\sim[\beta \subseteq \alpha]$
(4)	$\alpha \supseteq \beta$	$=_{df}$	$\beta \subseteq \alpha$
(5)	$\alpha \supset \beta$	$=_{df}$	$\beta \subset \alpha$
(6)	$\{\nu : \mathbb{F}\}$	$=_{df}$	$\iota Sx \forall y\{y \in x \leftrightarrow \mathbb{F}\}\}$
(7)	{α}	$=_{df}$	{x: x=α}
(8)	{α,β}	$=_{df}$	$\{x: x=\alpha \vee x=\beta\}$
(9)	{α,β,γ}	$=_{df}$	$\{x: x=\alpha \vee x=\beta \vee x=\gamma\}$
	etc.		
(10)	$\alpha \cap \beta$	$=_{df}$	$\{x: x \in \alpha$ & $x \in \beta\}$
(11)	$\alpha \cup \beta$	$=_{df}$	$\{x: x \in \alpha \vee x \in \beta\}$
(12)	$\alpha - \beta$	$=_{df}$	$\{x: x \in \alpha$ & $x \notin \beta\}$
(13)	α(β)	$=_{df}$	app(α,β)
(14)	Dom[α,β]	$=_{df}$	Fun[β] & $\exists x[x = \beta(\alpha)]$
(15)	Ran[α,β]	$=_{df}$	Fun[β] & $\exists x[\alpha = \beta(x)]$
(16)	dom(α)	$=_{df}$	{x : Dom[x,α]}
(17)	ran(α)	$=_{df}$	{x : Ran[x,α]}
(18)	$\alpha:\beta\rightarrow\gamma$	$=_{df}$	Fun[α] & dom(α) = β & ran(α) $\subseteq$ γ
(19)	$\alpha:\beta\rightarrow\gamma$ (onto)	$=_{df}$	Fun[α] & dom(α) = β & ran(α) = γ
(20)	$\alpha:\beta\rightarrow\gamma$ (1-1)	$=_{df}$	Fun[α] & dom(α) = β & ran(α) = γ & $\forall xy\{x,y \in \beta \rightarrow. \alpha(x)=\alpha(y) \rightarrow x=y\}$
(21)	$\alpha:\beta\leftrightarrow\gamma$	$=_{df}$	$\alpha:\beta\rightarrow\gamma$ (onto) & $\alpha:\beta\rightarrow\gamma$ (1-1)

5. Primitive Theses (Axioms)

(1) $\text{Set}[\alpha] \rightarrow \exists x[x = \alpha]$

(2) $\text{Fun}[\alpha] \rightarrow \exists x[x = \alpha]$

(3) $\forall xy\{x\in y \rightarrow \text{Set}[y]\}$

(4) $\forall xyz\{x(y)=z \rightarrow \text{Fun}[x]\}$

(5) $\exists s\forall x\{\text{Uhr}[x] \leftrightarrow x\in s\}$

(6) $\text{Set}[\varnothing]$

(7) $\sim\exists x[x\in\varnothing]$

(8) $\sim\text{Fun}[\varnothing]$

(9) $\forall s_1s_2\{\forall x(x\in s_1 \leftrightarrow x\in s_2) .\rightarrow s_1=s_2\}$

(10) $[\forall\mathbb{P}]\ \forall s_1\exists s_2\forall x\{x\in s_2 \leftrightarrow. x\in s_1 \ \&\ \mathbb{P}[x]\}$

(11) $\forall s_1s_2\exists s_3\{s_1 \subseteq s_3 \ \ \&\ \ s_2 \subseteq s_3\}$

(12) $\forall f\ \exists y\{y\neq\varnothing \ \ \&\ \ y=\text{dom}(f)\}$

(13) $\forall f\ \exists y[y = \text{ran}(f)]\}$

(14) $\forall f_1\forall f_2\{\text{dom}(f_1)=\text{dom}(f_2)\ \&\ \forall x\{x\in \text{dom}(f_1) \rightarrow [f_1(x)=f_2(x)]\} .\rightarrow f_1=f_2\}$

(15) $[\forall\mathbb{R}]\ \forall s_1\forall s_2\{\forall x\in s_1\exists!y\in s_2\mathbb{R}[x,y] \rightarrow \exists f\forall xy\{f(x)=y \leftrightarrow \mathbb{R}[x,y]\}\}$

(16) $[\forall\mathbb{R}]\ \forall s_1\forall s_2\{\forall x\in s_1\exists y\in s_2\mathbb{R}[x,y] \rightarrow \exists[f:s_1\rightarrow s_2]\forall xy\{f(x)=y \rightarrow \mathbb{R}[x,y]\}\}$

6. Examples of Derivative Theses (Theorems)

(1) $\exists x\{Set[x] \& \sim Fun[x]\}$

(2) $\forall s\{\sim\exists y[y\in s] \rightarrow x=\varnothing\}$

(3) $\forall s\{s\neq\varnothing \rightarrow \exists y[y\in s]\}$

(4) $[\forall \mathbb{P}]\ \forall x\{\mathbb{P}[x] \rightarrow Uhr[x]\} \rightarrow \exists s\{s = \{x : \mathbb{P}[x]\}\}$

(5) $\forall x\{Uhr[x] \rightarrow \exists s[s=\{x\}]\}$

(6) $\forall x\{Uhr[x] \& Uhr[y] .\!\rightarrow \exists s[s=\{x,y\}]\}$

etc.

(7) $\forall s_1 s_2 \exists s_3[s_3 = s_1 \cap s_2]$

(8) $\forall s_1 s_2 \exists s_3[s_3 = s_1 \cup s_2]$

(9) $\forall s_1 s_2 \exists s_3[s_3 = s_1 - s_2]$

(10) $\forall f\ \exists y[y\neq\varnothing \ \&\ y=ran(f)]\}$

(11) $\forall x \forall y\{Dom[x,y] \leftrightarrow x\in dom(y)\}$

(12) $\forall x \forall y\{Ran[x,y] \leftrightarrow x\in ran(y)\}$