1. Number-Words

By a *number-word*, or *numeral*, we mean a word (or word-like compound) that denotes a number.[1] In English, numerals *appear* to be used as quantifiers, as in the following examples.

> **two** dogs are barking
> Jay owns **three** dogs
> there are **four** dogs in the yard

On the other hand, numerals also figure in the following sorts of constructions.

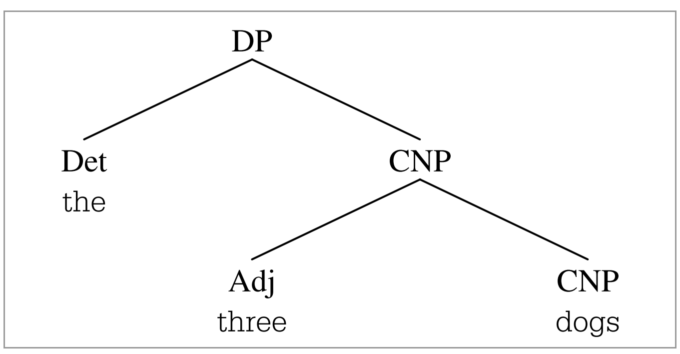
> the **three** dogs
> my **four** dogs
> all **five** dogs
> no **two** dogs are exactly alike
> we **three** kings of Orient are (bearing gifts we traverse afar…)

If numerals are quantifiers, then the first four phrases violate the prohibition against double-determiners. In the last phrase, the appositive ‘three kings’ cannot be replaced by QPs such as ‘some kings’, ‘most kings’, or ‘all kings’, but it can be replaced by ‘kings’.

2. Proposal – Numerals are Plural Adjectives

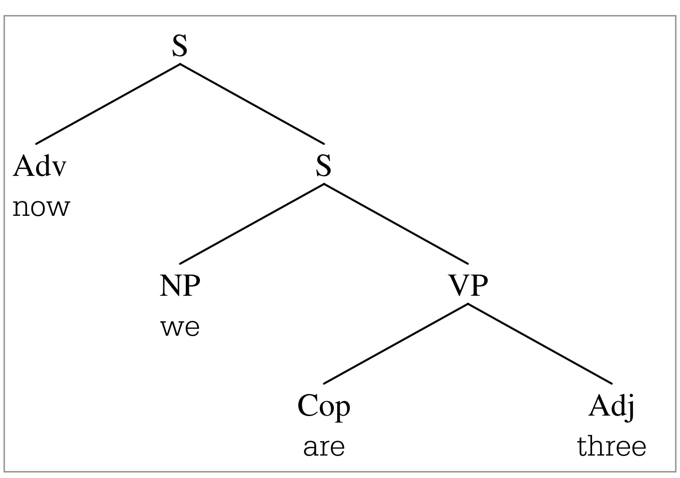
Given the latter data, we propose that numerals are fundamentally, *not* quantifiers, but are rather adjectives,[2] as illustrated in the following tree.

the three dogs



Here ‘three’ serves as a CNP-modifier, which is the principal role of adjectives. Numerals can also be used as bare-adjectives, although it is less common, as in the following poetic announcement of the arrival of a couple's first child.

now we are three



[1] Numerals are sometimes regarded as a species of number words – being logograms like ‘2’ rather than phonograms like ‘two’. Since this is not a semantically relevant distinction, we will simply use the terms interchangeably.
[2] This is in exact agreement with traditional lexicography.

3. Numerical-Adjectives

First, we propose the following lexical entries, which treat numerals as bare-adjectives.[3]

$[\![\text{one}]\!] \;=\; \lambda x_0 \mathbf{1}x$
$[\![\text{two}]\!] \;=\; \lambda x_0 \mathbf{2}x$
etc.

Here the numerical-predicates in the meta-language (alternatively, target-language) are understood as follows.

$\mathbf{1}[\alpha]$ =: α is a unit-entity
$\mathbf{2}[\alpha]$ =: α is a plurality consisting of two unit-entities
$\mathbf{3}[\alpha]$ =: α is a plurality consisting three unit-entities
etc.

The notion of 'unit' is primitive; unit-entities correspond to individuals in standard elementary first-order logic. On the other hand, 'consists of' is defined as follows.

$$\alpha \text{ consists of } \beta_1, \beta_2, \ldots, \beta_k \quad =_{\text{df}} \quad \alpha = \beta_1 \oplus \beta_2 \oplus \ldots \oplus \beta_k$$

Now, every bare-adjective gives rise to a CNP-modifier when combined with the [mod] operator, as illustrated in the following example.

1. those are three dogs

those	[+1]	are	three	[mod]	dog	s [plural]	
δ	$\lambda x\{x_1\}$		$\lambda x_0 \mathbf{3}x$	$\lambda Q_0 \lambda P_0 \lambda x_0\{Px \& Qx\}$	$\lambda x_0 \mathbf{D}x$	$\lambda P_0 \lambda x_0\{Px \& \mathbb{P}x\}$	
			$\lambda P_0 \lambda x_0\{Px \;\&\; \mathbf{3}x\}$		$\lambda x_0\{\mathbf{D}x \;\&\; \mathbb{P}x\}$		①
		$\lambda P_0\{P_1\}$	$\lambda x_0\{\mathbf{D}x \;\&\; \mathbf{3}x\}$				②
δ_1		$\lambda x_1\{\mathbf{D}x \;\&\; \mathbf{3}x\}$					
$\mathbf{D}\delta \;\&\; \mathbf{3}\delta$							

Note that ① involves applying the plural modifier $[\![\text{plural}]\!]$ to $[\![\text{dog}]\!]$, the latter of which consists of all dog-entities, including dog-individuals, dog-pluralities, and dog-masses (in principle). Also, note that the predicate $\mathbb{P}$ becomes redundant when combined with the **3**-predicate to produce ②, since we have the semantic principle that $\mathbf{3}x$ entails $\mathbb{P}x$. The output node says that δ is a dog-entity and δ consists of three elements, which amounts to saying that δ is a three-element plurality consisting of dogs, what we might call a 'dog-trio'.

4. Numerical Quantifiers

In the following examples, 'three dogs' behaves like a QP.

1. three dogs are barking
2. Jay owns three dogs

As before, when a CNP serves as an NP, we invoke the transformation that converts a CNP

[3] The morphology of number-words is taken for granted here. That is a topic unto itself, which is dealt with in the chapter "The Morphology of Number Words".

[type $D_0 \rightarrow S$] into an entity-sum [type ΣD]. Using this technique on the above examples, we get the following analyses.

three dogs are barking

three [mod]	dogs	[+1]	are barking
$\lambda P_0 \lambda x_0\{Px \& \mathbf{3}x\}$	$\lambda x_0\{\mathbf{D}x \& \mathbb{P}x\}$		
$\lambda x_0\{\mathbf{D}x \ \& \ \mathbf{3}x\}$			
$\Sigma\{\ x \mid \mathbf{D}x \ \& \ \mathbf{3}x\ \}$		$\lambda x\{x_1\}$	
$\Sigma\{\ x_1 \mid \mathbf{D}x \ \& \ \mathbf{3}x\ \}$			$\lambda x_1 \mathbf{B}x$
$\Sigma\{\ \mathbf{B}x \mid \mathbf{D}x \ \& \ \mathbf{3}x\ \}$			
$\exists x\ \{\ \mathbf{D}x \ \& \ \mathbf{3}x \ \& \ \mathbf{B}x\ \}$			

3. Jay owns three dogs

Jay [+1]	owns	three [mod]	dogs	[+2]
		$\lambda P_0 \lambda x_0\{Px \& \mathbf{3}x\}$	$\lambda x_0\{\mathbf{D}x \ \& \ \mathbb{P}x\}$	
		$\lambda x_0\{\mathbf{3}x \ \& \ \mathbf{D}x\}$		
		$\Sigma\{\ x \mid \mathbf{3}x \ \& \ \mathbf{D}x\ \}$		$\lambda x\{x_2\}$
	$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\Sigma\{\ y_2 \mid \mathbf{3}y \ \& \ \mathbf{D}y\ \}$		
J_1	$\Sigma\{\ \lambda x_1 \mathbf{O}xy \mid \mathbf{3}y \ \& \ \mathbf{D}y\ \}$			
$\Sigma\{\ \mathbf{O}Jy \mid \mathbf{3}y \ \& \ \mathbf{D}y\ \}$				
$\exists y\ \{\ \mathbf{3}y \ \& \ \mathbf{D}y \ \& \ \mathbf{O}Jy\ \}$				

This account also works when combined with logical operators, as in the following example.

4. Jay does not own three dogs

Jay [+1]	does-not	own	three dogs	[+2]	
			$\lambda y_0\{\mathbf{D}y \ \& \ \mathbf{3}y\}$		
			$\Sigma\{\ y \mid \mathbf{D}y \ \& \ \mathbf{3}y\ \}$	$\lambda y\{y_2\}$	
		$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\Sigma\{\ y_2 \mid \mathbf{D}y \ \& \ \mathbf{3}y\ \}$		
	$\lambda P_1 \lambda x_1 {\sim} Px$	$\Sigma\{\ \lambda x_1 \mathbf{O}xy \mid \mathbf{D}y \ \& \ \mathbf{3}y\ \}$			①
J_1	$\Pi\{\ \lambda x_1 {\sim} \mathbf{O}xy \mid \mathbf{D}y \ \& \ \mathbf{3}y\ \}$				②
$\Pi\{\ {\sim}\mathbf{O}Jy \mid \mathbf{D}y \ \& \ \mathbf{3}y\ \}$					
$\forall y\ \{\ \mathbf{D}y \ \& \ \mathbf{3}y \ .\!\rightarrow {\sim}\mathbf{O}Jy\ \}$					

Notice, in the move from ① to ②, that ⟦does not⟧ being anti-tonic converts Σ to Π, so that the sentence says that Jay does not own *any* plurality consisting of three dogs. The following is an alternative derivation in which the Σ-junction stays narrow, but produces a logically-equivalent result.

5. Jay does not own three dogs [alternative computation]

Jay [+1]	does not	own three dogs [+2]
		$\Sigma\{\lambda x_1 \mathbf{O}xy \mid \mathbf{D}y \ \&\ \mathbf{3}y\ \}$
	$\lambda P_1 \lambda x_1 {\sim} Px$	$\lambda x_1\ \exists y\{\mathbf{D}y \ \&\ \mathbf{3}y \ \&\ \mathbf{O}xy\}$
J_1	$\lambda x_1\ {\sim}\exists y\{\mathbf{D}y \ \&\ \mathbf{3}y \ \&\ \mathbf{O}xy\}$	
$\sim\exists y\ \{\ \mathbf{3}y \ \&\ \mathbf{D}y \ \&\ \mathbf{O}Jy\ \}$		

The following example illustrates how numerical-quantifiers interact with 'every'.

6. every man owns three dogs

every man [+1]	owns	three dogs [+2]
	$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\Sigma\{\ y_2 \mid \mathbf{D}y \ \&\ \mathbf{3}y\ \}$
$\wedge\{\ x_1 \mid \mathbf{M}x\ \}$	$\Sigma\{\ \lambda x_1 \mathbf{O}xy \mid \mathbf{D}y \ \&\ \mathbf{3}y\ \}$	
$\wedge\{\ \Sigma\{\ \mathbf{O}xy \mid \mathbf{D}y \ \&\ \mathbf{3}y\ \} \mid \mathbf{M}x\ \}$		
$\forall x\ \{\ \mathbf{M}x \rightarrow \exists y\ \{\ \mathbf{D}y \ \&\ \mathbf{3}y \ \&\ \mathbf{O}xy\ \}\ \}$		

5. A Complication

The above account of numerical quantifiers seems nice and tidy, but it is not the whole story. Suppose I ask you how many dogs you own, and you answer as follows,

(A) I own three dogs

I should be entitled to infer that you don't own four dogs. On the other hand, according to the account of 'three' given above, (A) is true precisely if the speaker owns at least one three-membered dog-plurality. But, presuming that dog-ownership is distributive,[4] if the speaker owns a four-membered dog-plurality, then *a fortiori* the speaker also owns four three-membered dog *sub*-pluralities.[5]

The question then is whether my inference is **pragmatic** or (purely) **semantic**. On pragmatic grounds, your answer *should* be *maximally informative*, so if you own four dogs, saying you own three dogs is truthful, but not maximally informative; indeed, it seems misleading. It is a violation, not of semantics, but of pragmatics.[6]

For this reason, English affords us more precise language by which to communicate, so that pragmatics plays a less critical role. In particular, in order to avoid confusion you might say

(A_2) I own **exactly** three dogs.

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[4] In other words, if one owns a plurality of dogs, then one owns every sub-plurality of those dogs.

[5] Let $\{d_1, d_2, d_3, d_4\}$ be a set of four dogs; then the following are four subsets consisting of three dogs each – $\{d_1, d_2, d_3\}$, $\{d_1, d_2, d_4\}$, $\{d_1, d_3, d_4\}$, $\{d_2, d_3, d_4\}$.

[6] Similarly, if I tell you I will pay you on Tuesday or Wednesday, when in fact I plan to pay you on Tuesday seems misleading, although it is true.

or perhaps

(A_3) I own **at least** three dogs.

Now, whereas 'three' by itself behaves more like an adjective, the latter two phrases 'exactly three' and 'at least three' behave more like quantifiers. For example, the following phrases

the **three** dogs
my **four** dogs
all **five** dogs
no **two** dogs are exactly alike
we **three** kings

become considerably less felicitous when we insert 'exactly' or 'at least', as seen in the following examples.

the **at least three** dogs
my **exactly four** dogs
all **at least five** dogs
no **exactly two** dogs are exactly alike
we **at least three** kings …

We accordingly need an account of 'exactly' and 'at least' when they attach to basic number-words.[7]

6. Second-Order Numerical-Predicates

Thus far, we use numerals in the meta-language as first-order predicates [D→S], informally characterized as follows.

1[α] =: α is one
2[α] =: α are two
etc.

We now propose yet another use of numerals in the meta-language, according to which they are second-order predicates [type (D→S)→S], characterized as follows.

1[P] =: there is exactly one item that has property P
$=_{df}$ $\exists x \forall y \{ Py \leftrightarrow y = x \}$

2[P] =: there are exactly two items that have property P
$=_{df}$ $\exists x_1 \exists x_2 \{ x_1 \neq x_2 \ \& \ \forall y \{ Py \leftrightarrow . \ y = x_1 \vee y = x_2 \} \}$

3[P] =: there are exactly three items that have property P
$=_{df}$ $\exists x_1 \exists x_2 x_3 \{ x_1 \neq x_2 \ \& \ x_1 \neq x_3 \ \& \ x_2 \neq x_3 \ \&$
$\forall y \{ Py \leftrightarrow . \ y = x_1 \vee y = x_2 \vee y = x_3 \} \}$
etc.

We also introduce the following natural shorthand as follows,

[7] By which I mean the words and word-like compounds children proudly recite – including, for example:
one, fifty, two hundred, three thousand sixty-five,
six million three hundred thousand one hundred twenty five

$$\mathbf{N}\nu\Phi \quad =_{df} \quad \mathbf{N}[\lambda\nu\Phi]$$

where **N** is a numerical-predicate (as above), ν is a variable, and Φ is a formula.

With these notions in hand, we next offer the following semantic account of 'exactly'.[8]

$$[\![\text{exactly}]\!] \quad = \quad \lambda\mathbf{N}\lambda P_0\lambda Q.\mathbf{N}x\{Px\&Qx\}$$

Here, the letter '**N**' serves as a variable ranging over first-order numerical-predicates (type D→S). When applied to particular number-words, we have the following.

$$[\![\text{exactly one}]\!] \quad = \quad \lambda P_0\lambda Q.\mathbf{1}x\{Px\&Qx\}$$
$$[\![\text{exactly two}]\!] \quad = \quad \lambda P_0\lambda Q.\mathbf{2}x\{Px\&Qx\}$$
etc.

The following is an example calculation.

1. Jay owns exactly two dogs

<table>
<tr><th>Jay [+1]</th><th>owns</th><th>exactly</th><th>two</th><th>dogs</th><th>[+2]</th></tr>
<tr><td rowspan="4"></td><td rowspan="3"></td><td>$\lambda\mathbf{N}\lambda P_0\lambda Q.\mathbb{N}x\{Px\&Qx\}$</td><td>$\mathbf{2}$</td><td rowspan="2">$\mathbf{D}_0$</td><td rowspan="3">$\lambda x\{x_2\}$</td></tr>
<tr><td colspan="2">$\lambda P_0\lambda Q.\mathbf{2}x\{Px\&Qx\}$</td></tr>
<tr><td colspan="3">$\lambda Q.\mathbf{2}x\{\mathbf{D}x\&Qx\}$</td></tr>
<tr><td>$\lambda y_2\lambda x_1\mathbf{O}xy$</td><td colspan="4">$\lambda Q_2.\mathbf{2}x\{\mathbf{D}x\&Qx\}$</td></tr>
<tr><td>J_1</td><td colspan="5">$\lambda x_1\ \mathbf{2}y\{\mathbf{D}y\ \&\ \mathbf{O}xy\}$</td></tr>
<tr><td colspan="6">$\mathbf{2}y\{\mathbf{D}y\ \&\ \mathbf{O}Jy\}$</td></tr>
</table>

7. Junctions

Earlier, we saw how to analyze quantifiers in terms of infinitary-operators (junctions). One is naturally inclined to ask whether such an analysis is available for numerical-quantifiers. The answer is "yes", which is spelled out as follows, which becomes our official semantic analysis.

First, we introduce an infinitary-operator $\boxtimes$, which corresponds to (anadic) **exclusive-or**, which has the following semantic property.

$\boxtimes\{P_1, \ldots, P_k\}$ is true

iff

exactly one of $\{P_1, \ldots, P_k\}$ is true

We next offer the following alternative analysis of 'exactly'.

$$[\![\text{exactly}]\!] \quad = \quad \lambda\mathbf{N}\lambda P_0 \boxtimes\{x \mid \mathbf{N}x\ \&\ Px\}$$

For example,

[8] This account only considers the use of 'exactly' in combination with basic number words.

(1) ⟦exactly one dog⟧ = $\boxtimes\{x \mid \mathbf{1}x \;\&\; \mathbf{D}x\}$

(2) ⟦exactly two dogs⟧ = $\boxtimes\{x \mid \mathbf{2}x \;\&\; \mathbf{D}x\}$

etc.

where (1) amounts to

$$\text{dog}_1 \boxtimes \text{dog}_2 \boxtimes \ldots \boxtimes \text{dog}_k$$

where $\{\text{dog}_1, \text{dog}_2, \ldots, \text{dog}_k\}$ is the set of all *individual* dogs, and $\boxtimes$ is the exclusive-or operator,[9] and (2) amounts to

$$\text{dog-duo}_1 \boxtimes \text{dog-duo}_2 \boxtimes \ldots \boxtimes \text{dog-duo}_k$$

where $\{\text{dog-duo}_1, \text{dog-duo}_2, \ldots, \text{dog-duo}_k\}$ is the set of all dog-duos (i.e., pluralities consisting of two dogs).

The following is an example calculation.

2. Jay owns exactly-two dogs

Jay [+1]	owns	exactly	two	dogs	[+2]	
		$\lambda\mathbf{N}\lambda P_0\boxtimes\{y \mid Py \;\&\; \mathbf{N}y\}$	$\mathbf{2}$			
		$\lambda P_0\boxtimes\{y \mid Py \;\&\; \mathbf{2}y\}$		$\mathbf{D}_0$		
		$\boxtimes\{ y \mid \mathbf{D}y \;\&\; \mathbf{2}y \}$			$\lambda x\{x_2\}$	
	$\lambda y_2\lambda x_1\mathbf{O}xy$	$\boxtimes\{ y_2 \mid \mathbf{D}y \;\&\; \mathbf{2}y \}$				
J_1	$\boxtimes\{ \lambda x_1\mathbf{O}xy \mid \mathbf{D}y \;\&\; \mathbf{2}y \}$					
$\boxtimes\{ \mathbf{O}\text{J}y \mid \mathbf{D}y \;\&\; \mathbf{2}y \}$						①
$\exists! y\{ \mathbf{D}y \;\&\; \mathbf{2}y \;\&\; \mathbf{O}\text{J}y \}$						②
$\mathbf{2}y\{\mathbf{D}y \;\&\; \mathbf{O}\text{J}y\}$						③

Line ① amounts to the following.

$$\mathbf{O}[\text{J}, \text{dog-duo}_1] \boxtimes \ldots \boxtimes \mathbf{O}[\text{J}, \text{dog-duo}_k]$$

From this it follows that there is exactly one dog-duo that Jay owns, which is what ② says using the familiar logical notation '∃!'. This move is underwritten by the following further type-logical principle.

$\boxtimes_{\mathrm{v}}\{\Psi \mid \Phi\}$ / $\exists!\mathrm{v}\{\Phi \;\&\; \Psi\}$	[⊠-simplification]

Here, as usual, we have the following first-order definition.

$$\exists!\mathbf{v}\Phi \quad =_{\text{df}} \quad \exists\mathbf{w}\forall\mathbf{v}\{\Phi \leftrightarrow \mathbf{v}=\mathbf{w}\}$$

Here, **v** is any variable, and **w** is any variable distinct from **v** and not free in Φ.

[9] Notwithstanding the formatting of ⊠, it is not a binary-operator, but rather an anadic-operator which is formatted so that a "copy" is placed between each argument and the next. It is well-known that reducing this anadic-operator to a dyadic exclusive-or operator produces undesirable (ridiculous) results.

8. Scope Ambiguity

When quantifier-phrases are combined, we often get scope-ambiguities, and numerical-quantifiers are no exception, as illustrated in the following example.

exactly-two men respect exactly-two women

Indeed, this has **three** readings, since it can be used to answer three quite different questions.[10]

1. how many men respect exactly-two women?
2. how many women are respected by exactly-two men?
3. how many men respect how many women?

The first two are computed respectively as follows.

1. exactly two men respect exactly two women ['exactly two men' has wide scope]

exactly two men [+1]	respect	exactly two woman [+2]	
	$\lambda y_2 \lambda x_1 \mathbf{R}xy$	$\boxtimes\{\, y_2 \mid \mathbf{W}y \,\&\, \mathbf{2}y$	
$\boxtimes\{\, x_1 \mid \mathbf{M}x \,\&\, \mathbf{2}x \,\}$	$\boxtimes\{\, \lambda x_1 \mathbf{R}xy \mid \mathbf{W}y \,\&\, \mathbf{2}y \,\}$		①
$\boxtimes\{\, \{\, \mathbf{R}xy \mid \mathbf{W}y \,\&\, \mathbf{2}y \,\} \mid \mathbf{M}x \,\&\, \mathbf{2}x \,\}$			②
$\mathbf{2}x\ \{\, \mathbf{M}x \,\&\, \mathbf{2}y\{\mathbf{W}y \,\&\, \mathbf{R}xy\}\}$			

Here, in deriving ② from ①, the first junction absorbs the second junction. By contrast, in the following, the second junction absorbs the first junction.

2. exactly two men respect exactly two women ['exactly two women' has wide scope]

exactly two men [+1]	respect	exactly two woman [+2]	
	$\lambda y_2 \lambda x_1 \mathbf{R}xy$	$\boxtimes\{\, y_2 \mid \mathbf{W}y \,\&\, \mathbf{2}y$	
$\boxtimes\{\, x_1 \mid \mathbf{M}x \,\&\, \mathbf{2}x \,\}$	$\boxtimes\{\, \lambda x_1 \mathbf{R}xy \mid \mathbf{W}y \,\&\, \mathbf{2}y \,\}$		①
$\boxtimes\{\, \boxtimes\{\, \mathbf{R}xy \mid \mathbf{M}x \,\&\, \mathbf{2}x \,\} \mid \mathbf{W}y \,\&\, \mathbf{2}y \,\}$			②
$\mathbf{2}y\ \{\, \mathbf{W}y \,\&\, \mathbf{2}x\{\mathbf{M}x \,\&\, \mathbf{R}xy\}\,\}$			

The third reading is considerably more complicated and corresponds to a non-linear reading of the quantifier-phrases, according to which neither gains wide-scope over the other; rather, the quantifier phrases act in parallel.[11] Coming up with a type-theory formula that conveys this reading is not difficult.

3. $2x\{\mathbf{M}x \,\&\, \exists y\{\mathbf{W}y \,\&\, \mathbf{R}xy\}\} \,\&\, 2y\{\mathbf{W}y \,\&\, \exists x\{\mathbf{M}x \,\&\, \mathbf{R}xy\}\}$

The issue, as usual, is compositionality – how do we *derive* the proposed semantic-value from the semantic-values of the morphemes?

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[10] We presume a simple-minded account of *respect* according to which it stands between individuals and not groups of individuals. Otherwise, there are further readings as well. Also, 'respect' is stative; if we used an eventive verb, there also would be other readings, according to which we are counting events, as in 'three ships passed through three canals'.

[11] This is also known as the *cumulative reading* of the quantifiers. See, for example, Remko Scha (1981), 'Distributive, collective, and cumulative quantification', *Formal methods in the study of language*, ed. T. Janssen and M. Stokhof.