1. Numerical Adjectives

We treat number-words as modifier-adjectives [not bare-adjectives], semantically rendered as follows

$[\![\text{one}]\!] = \lambda P_0 \lambda x_0\, \mathbf{1}(P)[x]$
$[\![\text{two}]\!] = \lambda P_0 \lambda x_0\, \mathbf{2}(P)[x]$
etc.

where the numerical expressions in the meta-language (alternatively, target-language) are understood as follows.

$\mathbf{1}(P)[\alpha]$	=:	α is one P
$\mathbf{2}(P)[\alpha]$	=:	α is/are[1] two P
$\mathbf{3}(P)[\alpha]$	=:	α is/are three P
etc.		

What counts as one P – for example, one person, one electron, one acre, or one mile – is taken to be primitive. On the other hand, what counts as two P, three P, etc., is logically constrained by the following.

Fundamental Measurement Principle

$(m+n)(P)[\alpha]$ iff $\exists x \exists y \{ x \perp y \ \&\ \alpha = x \oplus y \ \&\ m(P)[x] \ \&\ n(P)[y] \}$

Here, α, x, and y are entities, $\perp$ is mereological-disjointness, and $\oplus$ is mereological-sum. Also, m and n are natural numbers, if P measures count-objects, and m and n are rational numbers,[2] if P measures mass-objects.

Note carefully that mereological-disjointness ($\perp$) is a bit tricky, since there are at least three different ways in which part-whole ideas are used.

(1) natural-mereology		one individual is naturally part of another; e.g., a proton is part of a nucleus.
(2) class-mereology		one plurality (class) is part of another; e.g., the (class of) senators is part of the (class of) politicians. Disjointness is set-theoretic.
(3) mass-mereology		one portion of matter is part of another; e.g. the gold in a statue is part of (all) the gold.

So when we count body parts, we use class-mereology, not natural-mereology; so one's left-elbow and one's left-arm count as disjoint, and hence two things, even though one's left-elbow is a natural part of one's left-arm.

Still, it's not so cut-and-dried, since matter can make up individuals, as in gold and statues,[3] and individuals can make up matter as in gold-atoms and gold material. Also, what

[1] The use of 'is' versus 'are' is dictated by whether P is a count-measure-noun (e.g., 'persons') or a mass-measure-noun (e.g., 'acres'). The underlying difference between persons (or planets, or electrons) and acres (or calories, or gallons) is that the former, but not the latter, are "true" individuals. Counting persons is quite different from "counting" calories.
[2] Rational numbers suffice for actual measurements, but real numbers (rational numbers plus irrational numbers) are much easier to work with theoretically.

counts as two places? Places can be both individuals and masses. Amherst and Massachusetts are two places insofar as they are logically distinct. Does this mean we are in two places at the same time?

2. Example

1. Jay owns three dogs

Jay [+1]	owns	three	dogs	[+2]
		$\lambda P_0 \lambda x_0 \mathbf{3}(P)[x]$	$\lambda x_0\{\mathbf{D}x \,\&\, \mathbb{P}x\}$	
		$\lambda x_0\{\mathbf{3D}x\}$		
		$\Sigma\{\, y \mid \mathbf{3D}y \,\}$		$\lambda x\{x_2\}$
	$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\Sigma\{\, y_2 \mid \mathbf{3D}y \,\}$		
J_1	$\Sigma\{\, \lambda x_1 \mathbf{O}xy \mid \mathbf{3D}y \,\}$			
$\Sigma\{\, \mathbf{O}Jy \mid \mathbf{3D}y \,\}$				
$\exists y\, \{\, \mathbf{3D}y \,\&\, \mathbf{O}Jy \,\}$				

This reads ‘Jay owns three dogs’ as saying that there is an entity y such that y is a dog-trio and Jay owns y.

3. A Complication

The problem is that, according to this reading, if Jay owns four dogs, then he also owns three dogs. For, if he owns four dogs, he owns at least one dog-quartet, so he owns four dog-trios, so he owns at least one dog-trio, so he owns three dogs. On the other hand, it seems that if I ask you how many dogs Jay owns, and you answer “three”, I should be able to infer that he does not own four dogs. We take this to be a pragmatic implication based on the principle of maximality.

In order to minimize dependence on pragmatics, English provides more precise language. In particular, in order to avoid confusion you might say

(A_2) Jay owns **exactly** three dogs,

or (A_3) Jay owns **at least** three dogs.

4. Numerical Quantifiers

Whereas ‘one’, ‘two hundred’, etc., are best understood as adjectives rather than quantifiers, ‘exactly one’ and ‘at least two hundred’ are best understood as quantifiers, which we propose to analyze as follows.

$$[\![\text{at least}]\!] = \lambda \mathbf{N}_0\, \lambda P_0 \bigvee\{\, x \mid \mathbf{N}(P)[x] \,\}$$

$$[\![\text{exactly}]\!] = \lambda \mathbf{N}_0\, \lambda P_0 \boxtimes\{\, x \mid \mathbf{N}(P)[x] \,\}$$

[3] We maintain that a (pure) gold statue is logically distinct from the gold that makes up the statue. For, in particular, they don't have the same history. The statue came into existence a few (or few hundred) years ago, whereas the gold came into existence in a supernova explosion millions (perhaps billions) of years ago.

Here the variable '**N**' ranges over numerical-adjectives in the meta-language, $\vee$ is the disjunction operator, and $\boxtimes$ is the exclusive-disjunction operator, which is characterized by the following semantic principle.

$$\boxtimes\{P_1, \ldots, P_k\} \text{ is true} \quad \text{iff} \quad \text{exactly one of } \{P_1, \ldots, P_k\} \text{ is true}$$

5. Examples

1. Jay owns at least three dogs

Jay [+1]	owns	at–least	three	dogs	[+2]
		$\lambda\mathbf{N}_0\, \lambda P_0 \vee\{\, y \mid \mathbf{N}(P)[y] \,\}$	$\lambda P_0 \lambda x_0\, \mathbf{3}(P)[x]$		
		$\lambda P_0 \vee\{\, y \mid \mathbf{3}Py \,\}$		$\mathbf{D}_0$	
		$\vee\{\, y \mid \mathbf{3D}y \,\}$			$\lambda x\{x_2\}$
	$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\vee\{\, y_2 \mid \mathbf{3D}y \,\}$			
J_1	$\vee\{\, \lambda x_1 \mathbf{O}xy \mid \mathbf{3D}y \,\}$				
$\vee\{\, \mathbf{O}\mathrm{J}y \mid \mathbf{3D}y \,\}$					
$\exists y\, \{\, \mathbf{3D}y \,\&\, \mathbf{O}\mathrm{J}y \,\}$					

2. Jay owns exactly three dogs

Jay [+1]	owns	exactly	three	dogs	[+2]	
		$\lambda\mathbf{N}_0 \lambda P_0 \boxtimes\{\, y \mid \mathbf{N}(P)[y] \,\}$	$\mathbf{3}_0$			
		$\lambda P_0 \boxtimes\{\, y \mid \mathbf{3}Py \,\}$		$\mathbf{D}_0$		
		$\boxtimes\{\, y \mid \mathbf{3D}y \,\}$			$\lambda x\{x_2\}$	
	$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\boxtimes\{\, y_2 \mid \mathbf{3D}y \,\}$				
J_1	$\boxtimes\{\, \lambda x_1 \mathbf{O}xy \mid \mathbf{3D}y \,\}$					
$\boxtimes\{\, \mathbf{O}\mathrm{J}y \mid \mathbf{3D}y \,\}$						①
$\exists! y\{\, \mathbf{3D}y \,\&\, \mathbf{O}\mathrm{J}y \,\}$						②
$3y\{\mathbf{D}y \,\&\, \mathbf{O}\mathrm{J}y\}$						③

Line ① amounts to the following.

$$\mathbf{O}[\mathrm{J}, \text{dog-trio}_1] \boxtimes \ldots \boxtimes \mathbf{O}[\mathrm{J}, \text{dog-trio}_k]$$

Given the nature of exclusive-or, this amounts to saying that there is exactly one dog-trio that Jay owns. This is also what ② says, but using the familiar logical notation '∃!' ["there is exactly one…"]. The latter move is underwritten by the following type-logical principle.

$$\boxtimes_\nu\{\Psi \mid \Phi\} \;/\; \exists!\nu\{\Phi \,\&\, \Psi\} \qquad [\boxtimes\text{-simplification}]$$

Here, we have the following first-order definition.

$$\exists!\mathbf{v}\Phi \quad =_{df} \quad \exists\mathbf{w}\forall\mathbf{v}\{\Phi \leftrightarrow \mathbf{v}{=}\mathbf{w}\}$$

Here, **v** is any variable, and **w** is any variable that is distinct from **v** and not free in Φ.

Finally, line ③ is obtained from ② in accordance with notation and principles explained in the following section.

6. Second-Order Numerical-Predicates

Thus far, we use numerals in the meta-language as predicate-modifiers [type: D→S.→D→S], informally characterized as follows.

1(P)[α] =: α is one P
2(P)[α] =: α are two P
etc.

We can also use numerals in the meta-language as second-order predicates [type: (D→S)→S], characterized as follows.

$\mathbb{1}$[P] =: there is exactly one item that has property P
$=_{df}$ $\exists x \forall y \{ \mathrm{P}y \leftrightarrow y=x \}$

$\mathbb{2}$[P] =: there are exactly two items that have property P
$=_{df}$ $\exists x_1 \exists x_2 \{ x_1 \neq x_2 \; \& \; \forall y \{ \mathrm{P}y \leftrightarrow . \; y=x_1 \vee y=x_2 \} \}$

$\mathbb{3}$[P] =: there are exactly three items that have property P
$=_{df}$ $\exists x_1 \exists x_2 x_3 \{ x_1 \neq x_2 \; \& \; x_1 \neq x_3 \; \& \; x_2 \neq x_3 \; \& \; \forall y \{ \mathrm{P}y \leftrightarrow . \; y=x_1 \vee y=x_2 \vee y=x_3 \} \}$

etc.

We also introduce the following natural shorthand,

$\mathbb{N}\nu\Phi$ $=_{df}$ $\mathbb{N}[\lambda\nu\Phi]$

where ν is a variable, Φ is a formula, and $\mathbb{N}$ is a second-order numerical-predicate (as above).

Note carefully that the move from

$\exists!x\{$ **N**(P)x & Qx $\}$

to

$\mathbb{N}x\{\mathrm{P}x \; \& \; \mathrm{Q}x\}$

is only valid for count-entities. So, for example, the following is not a proper transformation.

Jay owns exactly-three acres of land
↷
$\exists!x\{$ **3**(**A**)x & **L**x & **O**J$x\}$ ✓
↷
3$x\{$**A**x & **L**x **O**J$x\}$ ※

The first formula correctly reads the original sentence as saying there is exactly one three-acre parcel of land that Jay owns. On the other hand, the latter says there are exactly three one-acre parcels of land that Jay owns. But if Jay owns three acres of land, he owns infinitely-many one-acre parcels of land (many of which overlap).

7. Scope Ambiguity

When quantifier-phrases are combined, we often get scope-ambiguities, and numerical-quantifiers are no exception, as illustrated in the following example.

exactly-two men respect exactly-two women

Indeed, this has **three** readings, since it can be used to answer three quite different questions.[4]

1. how many men respect exactly-two women?
2. how many women are respected by exactly-two men?
3. how many men respect how many women?

The first two are computed respectively as follows.

3. exactly two men respect exactly two women ['exactly two men' has wide scope]

exactly two men [+1]	respect	exactly two woman [+2]	
	$\lambda y_2 \lambda x_1 \mathbf{R}xy$	$\boxtimes\{\, y_2 \mid \mathbf{2W}y \,\}$	
$\boxtimes\{\, x_1 \mid \mathbf{2M}x \,\}$	$\boxtimes\{\, \lambda x_1 \mathbf{R}xy \mid \mathbf{W}y \;\&\; \mathbf{2}y \,\}$		①
$\boxtimes\{\, \boxtimes\{\, \mathbf{R}xy \mid \mathbf{2W}y \,\} \mid \mathbf{2M}x \,\}$			②
$\exists! x \{\, \mathbf{2M}x \;\&\; \exists! y\{\mathbf{2W}y \;\&\; \mathbf{R}xy\}\}$			
$2x\{\, \mathbf{M}x \;\&\; 2y\{\mathbf{W}y \;\&\; \mathbf{R}xy\} \,\}$			

Here, in deriving ② from ①, the first junction absorbs the second junction. By contrast, in the following, the second junction absorbs the first junction.

4. exactly two men respect exactly two women ['exactly two women' has wide scope]

exactly two men [+1]	respect	exactly two woman [+2]	
	$\lambda y_2 \lambda x_1 \mathbf{R}xy$	$\boxtimes\{\, y_2 \mid \mathbf{2W}y \,\}$	
$\boxtimes\{\, x_1 \mid \mathbf{M}x \;\&\; \mathbf{2}x \,\}$	$\boxtimes\{\, \lambda x_1 \mathbf{R}xy \mid \mathbf{2W}y \,\}$		①
$\boxtimes\{\, \boxtimes\{\, \mathbf{R}xy \mid \mathbf{2M}x \,\} \mid \mathbf{2W}y \,\}$			②
$\exists! y\{\, \mathbf{2W}y \;\&\; \exists! x\{\mathbf{2M}x \;\&\; \mathbf{R}xy\} \,\}$			
$2y\{\, \mathbf{W}y \;\&\; 2x\{\mathbf{M}x \;\&\; \mathbf{R}xy\} \,\}$			

The third reading is considerably more complicated and corresponds to a non-linear reading of the quantifier-phrases, according to which neither gains wide-scope over the other; rather, the quantifier-phrases act in parallel. Coming up with a logical formula that conveys this reading is not overly difficult. Here it is.

3. $2x\{\mathbf{M}x \;\&\; \exists y\{\mathbf{W}y \;\&\; \mathbf{R}xy\}\} \;\&\; 2y\{\mathbf{W}y \;\&\; \exists x\{\mathbf{M}x \;\&\; \mathbf{R}xy\}\}$

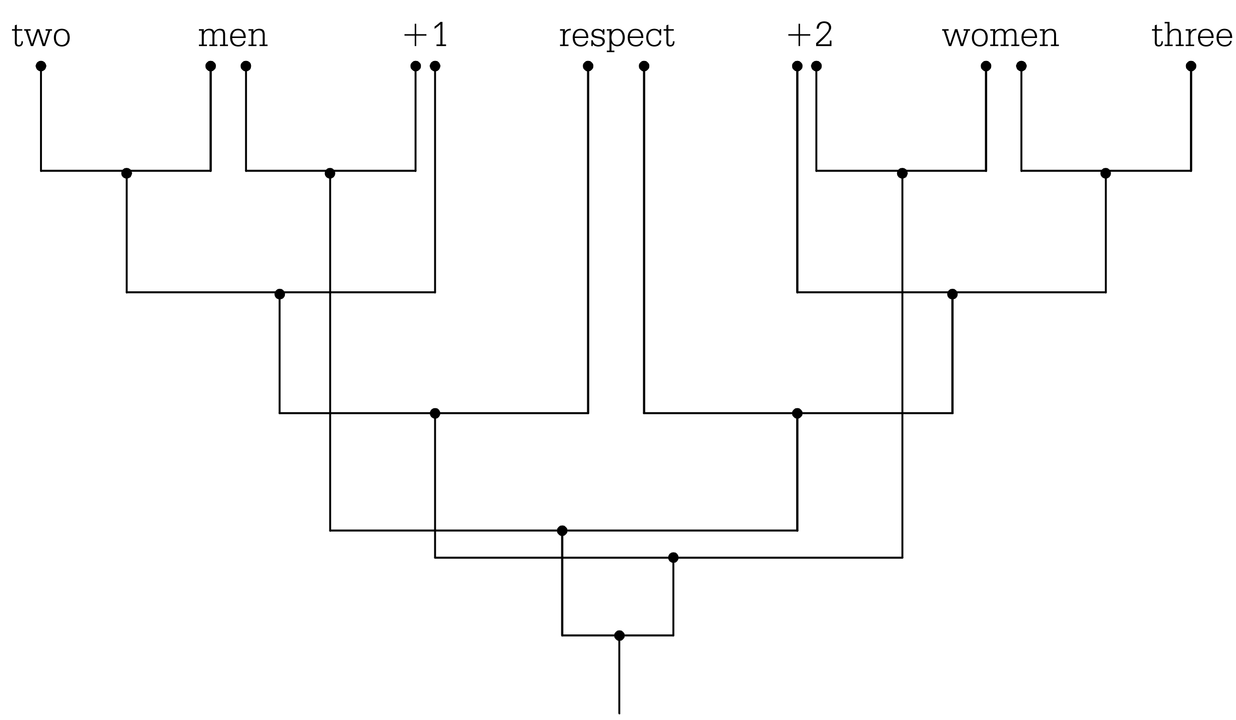
What is difficult, as usual, is to accomplish this compositionally – *deriving* the proposed semantic-value from the semantic-values of the morphemes.

[4] We presume a simple-minded account of *respect* according to which it stands between individuals and not groups of individuals. Otherwise, there are further (collective) readings as well. Also, 'respect' is stative. When eventive verbs and nouns are involved, counting is even more complicated. For example, what does the following mean? Last year, the world's busiest airport was Atlanta's airport (ATL), which served 90,039,280 passengers, which was followed by Chicago's airport (ORD), which served 69,353,654 passengers.

Our derivational system provides the answer as follows.

(1)	$\lambda P_0 \lambda Q\ 2x\{Px\ \&\ Qx\}$	$(D_0 \rightarrow S) \rightarrow [(D \rightarrow S) \rightarrow S]$	1	Pr
(2)	$\lambda x_0 \mathbf{M}x$	$D_0 \rightarrow S$	2	Pr
(3)	$\lambda x\{x_1\}$	$D \rightarrow D_1$	3	Pr
(4)	$\lambda y_2 \lambda x_1 \mathbf{R}xy$	$D_2 \rightarrow (D_1 \rightarrow S)$	4	Pr
(5)	$\lambda P_0 \lambda Q\ 2y\{Py\ \&\ Qy\}$	$(D_0 \rightarrow S) \rightarrow [(D \rightarrow S) \rightarrow S]$	5	Pr
(6)	$\lambda y_0 \mathbf{W}y$	$D_0 \rightarrow S$	6	Pr
(7)	$\lambda x\{x_2\}$	$D \rightarrow D_2$	7	Pr
(8)	$\lambda Q\ 2x\{\mathbf{M}x\ \&\ Qx\}$	$(D \rightarrow S) \rightarrow S$	12	1,2,λO
(9)	$\lambda Q_1\ 2x\{\mathbf{M}x\ \&\ Qx\}$	$(D_1 \rightarrow S) \rightarrow S$	123	3,8,ET
(10)	$\Sigma\{y \mid \mathbf{W}y\}$	ΣD	6	6,CN-conv
(11)	$\Sigma\{y_2 \mid \mathbf{W}y\}$	ΣD_2	67	7,10,Ж-rule
(12)	$\lambda x_1 \exists y\{\mathbf{W}y\ \&\ \mathbf{R}xy\}$	$D_1 \rightarrow S$	467	4,11,ET
(13)	$2x\{\mathbf{M}x\ \&\ \exists y\{\mathbf{W}y\ \&\ \mathbf{R}xy\}\}$	S	123467	9,12,λO
(14)	$\Sigma\{x \mid \mathbf{M}x\}$	ΣD	2	2,CN-conv
(15)	$\Sigma\{x_1 \mid \mathbf{M}x\}$	ΣD_1	23	3,14,λO
(16)	$\lambda y_2 \exists x\{\mathbf{M}x\ \&\ \mathbf{R}xy\}$	$D_2 \rightarrow S$	234	4,15,ET
(17)	$\lambda Q\ 2y\{\mathbf{W}y\ \&\ Qy\}$	$(D \rightarrow S) \rightarrow S$	56	5,6,λO
(18)	$\lambda Q_2\ 2y\{\mathbf{W}y\ \&\ Qy\}$	$(D_2 \rightarrow S) \rightarrow S$	567	7,17,λO
(17)	$2y\{\mathbf{W}y\ \&\ \exists x\{\mathbf{M}x\ \&\ \mathbf{R}xy\}\}$	S	234567	16,18,λO
(18)	$2x\{\mathbf{M}x\ \&\ \exists y\{\mathbf{W}y\ \&\ \mathbf{R}xy\}\} \times 2y\{\mathbf{W}y\ \&\ \exists x\{\mathbf{M}x\ \&\ \mathbf{R}xy\}\}$	$S \times S$	1234567	13,17,×I

Note that, in virtue of the non-linear relation between the two QPs, the derivation does not correspond to a tree; rather, it corresponds to a semi-lattice, drawn as follows, in which the morphemes are arranged so as to make the diagram neater.



8. Anaphoric-Pronouns connected to Numerical Quantifiers

What happens when we try to analyze the following variation on the "donkey" sentence.

1. if Jay owns exactly one dog, then it is a collie.

if	Jay [+1]	owns	exactly one dog [+2,–1]	then	(–1) it [+1]	is a collie
		$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\boxtimes\{\, y_2 \times y_{-1} \mid \mathbf{D}y \,\}$	$\varnothing$	$\lambda x_{-1}\{x_1\}$	$\lambda x_1 \mathbf{C}x$
	J_1	$\boxtimes\{\, \lambda x_1 \mathbf{O}xy \times y_{-1} \mid \mathbf{D}y \,\}$				
$\lambda P \lambda Q\{P \rightarrow Q\}$	$\boxtimes\{\, \mathbf{O}Jy \times y_{-1} \mid \mathbf{D}y \,\}$					
$\boxtimes\{\, \lambda Q\{\mathbf{O}Jy \rightarrow Q\} \times y_{-1} \mid \mathbf{D}y \,\}$				$\lambda x_{-1} \mathbf{C}x$		
$\boxtimes\{\, \lambda Q\{\mathbf{O}Jy \rightarrow Q\} \times \mathbf{C}x \mid \mathbf{D}y \,\}$						
$\boxtimes\{\, \mathbf{O}Jy \rightarrow \mathbf{C}x \mid \mathbf{D}y \,\}$						
$\exists! y\ \{\, \mathbf{D}y \ \&\ (\mathbf{O}Jy \rightarrow \mathbf{C}x) \,\}$						

Notice that we treat '**D**α' as short for '**1D**α', as in elementary logic, pretending we are dealing exclusively with singular-entities. This reads the sentence as saying that there is exactly one dog with the property that if Jay owns it then it is a collie. This is bizarre, and is surely not the intent of the original sentence. We can block this reading by insisting that ⊠ does not admit ⟦if⟧. But then we have no way to combine ⟦Jay owns exactly one dog⟧ with ⟦if⟧ unless we can somehow simplify the former.

The following derivation illustrates the proposed technique by which we perform the "simplification".

if	Jay [+1]	owns	exactly one dog [+2,–1]	then	(–1) it [+1]	is a collie
		$\lambda y_2 \lambda x_1 \mathbf{O}xy$	$\boxtimes\{\, y_2 \times y_{-1} \mid \mathbf{D}y \,\}$	$\varnothing$	$\lambda x_{-1}\{x_1\}$	$\lambda x_1 \mathbf{C}x$
	J_1	$\boxtimes\{\, \lambda x_1 \mathbf{O}xy \times y_{-1} \mid \mathbf{D}y \,\}$				
	$\boxtimes\{\, \mathbf{O}Jy \times y_{-1} \mid \mathbf{D}y \,\}$ ①					
	$\boxtimes\{\, \mathbf{O}Jy \mid \mathbf{D}y \,\} \times \iota y_{-1}\{\mathbf{D}y \ \&\ \mathbf{O}Jy\}$ ②					
$\lambda P \lambda Q\{P \rightarrow Q\}$	$\exists! y\{\, \mathbf{D}y \ \&\ \mathbf{O}Jy \,\} \times \iota y_{-1}\{\mathbf{D}y \ \&\ \mathbf{O}Jy\}$ ③					
$\lambda Q\{\, \exists! y\{\mathbf{D}y \ \&\ \mathbf{O}Jy\} \rightarrow Q \,\} \times \iota y_{-1}\{\mathbf{D}y \ \&\ \mathbf{O}Jy\}$				$\lambda x_{-1} \mathbf{C}x$		
$\lambda Q\{\, \exists! y\{\mathbf{D}y \ \&\ \mathbf{O}Jy\} \rightarrow Q \,\} \times \mathbf{C}[\iota y\{\mathbf{D}y \ \&\ \mathbf{O}Jy\}]$						
$\exists! y\{\mathbf{D}y \ \&\ \mathbf{O}Jy\} \rightarrow \mathbf{C}[\iota y\{\mathbf{D}y \ \&\ \mathbf{O}Jy\}]$						

First notice that ③ comes from ② by ⊠-simplification applied to the left-factor. More importantly, notice that ② come from ① by the new **anaphoric-exportation principle**, special to ⊠, given as follows.

$$\boxtimes_\nu\{\, \Psi \times \nu_\alpha \mid \Phi \,\} \quad / \quad \boxtimes_\nu\{\, \Psi \mid \Phi \,\} \times \iota\nu_\alpha\{\Phi \ \&\ \Psi\}$$

Here, Φ and Ψ are formulas, ν is a variable, and α is an anaphoric-marker.

So, when we say 'Jay owns exactly one dog' and we later use 'it' in reference to this sentence, we are cross-referencing a **tacit antecedent** – namely, 'the dog that Jay owns'.

9. Definite Determiner Phrases as Indefinite Noun Phrases – Russell's Theory of Descriptions

So far we have treated 'the' as having type C→D. As is fairly well known, Russell rejected this approach, proposing instead to treat 'the' more like a quantifier. As it turns out, the most cogent presentation of his views starts with the idea that 'the' is not a determiner of any sort, but is instead a CNP-modifier (*adjective*), just as 'a', 'one', 'two', etc., which means its type is C→C.

The following is our proposed analysis.

$$[\![\text{the}]\!] = \lambda P_0\{!P_0\}$$

where the exclamation point stands roughly for 'uniquely', and is formalized as follows.

$$!P =_{df} \lambda x \forall y\{Py \leftrightarrow y{=}x\}$$

The following is an example analysis.

1. the cat hates the dog

<table>
<tr><td>the</td><td>cat</td><td>[+1]</td><td>hates</td><td>the</td><td>dog</td><td>[+2]</td></tr>
<tr><td>$\lambda P_0\{!P_0\}$</td><td>$\mathbf{C}_0$</td><td rowspan="2"></td><td rowspan="3"></td><td>$\lambda P_0\{!P_0\}$</td><td>$\mathbf{D}_0$</td><td rowspan="2"></td></tr>
<tr><td colspan="2">$!\mathbf{C}_0$</td><td colspan="2">$!\mathbf{D}_0$</td></tr>
<tr><td colspan="2">$\Sigma\{\, x \mid !\mathbf{C}x \,\}$</td><td>$\lambda x\{x_1\}$</td><td colspan="2">$\Sigma\{\, y \mid !\mathbf{D}y \,\}$</td><td>$\lambda x\{x_2\}$</td></tr>
<tr><td colspan="3" rowspan="2">$\Sigma\{\, x_1 \mid !\mathbf{C}x \,\}$</td><td>$\lambda y_2 \lambda x_1 \mathbf{H}xy$</td><td colspan="3">$\Sigma\{\, y_2 \mid !\mathbf{D}y \,\}$</td></tr>
<tr><td colspan="4">$\Sigma\{\, \lambda x_1 \mathbf{H}xy \mid !\mathbf{D}y \,\}$</td></tr>
<tr><td colspan="7">$\Sigma\{\, \Sigma\{\, \mathbf{H}xy \mid !\mathbf{D}y \,\} \mid !\mathbf{C}x \,\}$</td></tr>
<tr><td colspan="7">$\exists x\{\, !\mathbf{C}x \;\&\; \exists y\{!\mathbf{D}y \;\&\; \mathbf{H}xy\} \,\}$</td></tr>
<tr><td colspan="7">$\exists x\{\, \forall z\{\mathbf{C}z \leftrightarrow z{=}x\} \;\&\; \exists y\{\, \forall z\{\mathbf{D}z \leftrightarrow z{=}y\} \;\&\; \mathbf{H}xy \,\} \,\}$ [5]</td></tr>
</table>

In other words, there is exactly one cat, and there is exactly one dog, and the former hates the latter.

This corresponds to Russell's analysis of definite descriptions.

There is still a problem for this account, and for our original account, of 'the'. The Russellian analysis of 'the dogs are (all) barking' would be:

there is exactly one dog-plurality, and its members are (all) barking

But if the dogs form a dog-trio, and they are barking, then there are three dog-duos that are barking, so there are four dog-pluralities barking, not exactly one. The concept of uniqueness (symbolized by '!') must be adjusted to take into account plural-nouns and mass-nouns.

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[5] Given the confounds of variable-binding, when we apply $!\mathbf{D}$ to 'y', we have to apply alphabetic-variance first. We do a similar maneuver with $!\mathbf{C}$ for symmetry.