Exercises on Monadic Lambda-Abstraction

1. Lambda-Conversion

The lambda-conversion rule for *monadic* lambda-abstracts can be stated as follows.

$$[\lambda \mathrm{v} \mathcal{E}]\langle\sigma\rangle = \mathcal{E}[\sigma/\mathrm{v}]$$

Here, $\mathcal{E}$ and σ are any expressions, v is any variable of the same type as σ, and $\mathcal{E}[\sigma/\mathrm{v}]$ results when σ replaces every free occurrence of v in $\mathcal{E}$, where v is **free for** σ in $\mathcal{E}$, which is to say that any variable that is free in σ is also free in $\mathcal{E}[\sigma/\mathrm{v}]$.

The identity symbol '=' is a functor of multi-type $(\mathfrak{I}\times\mathfrak{I})\rightarrow\mathrm{S}$, where $\mathfrak{I}$ is any type. If $\mathfrak{I} = \mathrm{S}$, then '=' corresponds to '$\leftrightarrow$'.

2. Alphabetic Variants

Because of the restrictions on substitution in λ-conversion, sometimes one cannot substitute the given expression Σ for the given expression Ω, but must first rewrite one or both of them as alphabetic variants. For example, '$\forall x\mathbf{F}x$' and '$\forall y\mathbf{F}y$' are alphabetic variants, but '$\mathbf{F}x$' and '$\mathbf{F}y$' are not.

3. Official Functor-Argument Notation

Officially, every (monadic) functor-argument compound is written as follows.

[functor]⟨argument⟩

However, we adopt variations and abbreviations in keeping with intermediate logic, as follows.

(1) Square-brackets may be deleted if the functor is simple.

(2) Corner-brackets may be deleted if the argument is simple, unless the functor is simple and has the same height as the argument.

(3) Corner-brackets may be replaced by round-parentheses or square-brackets in accordance with intermediate logic.

4. Bracket Shorthand (left association)

$$[\Lambda]\langle\alpha\rangle\langle\beta\rangle \quad =_{df} \quad [[\Lambda]\langle\alpha\rangle]\langle\beta\rangle$$

Examples,

$$[\lambda x\lambda y\mathbf{R}xy]\langle\mathbf{a}\rangle\langle\mathbf{b}\rangle \quad =_{df} \quad [[\lambda x\lambda y\mathbf{R}xy]\langle\mathbf{a}\rangle]\langle\mathbf{b}\rangle$$

$$[\lambda x\lambda y\lambda z\mathbf{R}xyz]\langle\mathbf{a}\rangle\langle\mathbf{b}\rangle\langle\mathbf{c}\rangle \quad =_{df} \quad [[[\lambda x\lambda y\lambda z\mathbf{R}xyz]\langle\mathbf{a}\rangle]\langle\mathbf{b}\rangle]\langle\mathbf{c}\rangle$$

5. Type Conventions

	un-bolded	bolded
lower-case Roman letters	variables of type D	proper-expressions of type D
upper-case Roman letters	variables of type D→S	proper-expressions of type D→S

6. Exercises

Evaluate the following expressions using lambda-conversion, showing all intermediate steps, placing each step on its own line.

(1) $[\lambda x \mathbf{R} x x]\langle \mathbf{a} \rangle$
(2) $[\lambda x \lambda y \mathbf{R} x y]\langle \mathbf{a} \rangle \langle \mathbf{b} \rangle$
(3) $[\lambda x \lambda y \lambda z \mathbf{R} x y z]\langle \mathbf{a} \rangle \langle \mathbf{b} \rangle \langle \mathbf{c} \rangle$
(4) $[\lambda x [\lambda y \mathbf{R} x y]\langle \mathbf{a} \rangle]\langle \mathbf{b} \rangle$
(5) $[\lambda x [\lambda y [\lambda z \mathbf{R} x y z]\langle \mathbf{a} \rangle]\langle \mathbf{b} \rangle]\langle \mathbf{c} \rangle$
(6) $[\lambda \mathrm{P} \forall x \mathrm{P} x]\langle \lambda x \mathbf{R} x x \rangle$
(7) $[\lambda \mathrm{P}\{\mathrm{P}\mathbf{a} \,\&\, \mathrm{P}\mathbf{b}\}]\langle \lambda x \mathbf{R} x \mathbf{c} \rangle$
(8) $[\lambda \mathrm{P} \lambda \mathrm{Q} \forall x \{\mathrm{P} x \rightarrow \mathrm{Q} x\}]\langle \lambda x \mathbf{R} x x \rangle \langle \lambda x \mathbf{R} x \mathbf{a} \rangle$
(9) $[\lambda \mathrm{P} \lambda x \forall y \{\mathrm{P} y \rightarrow \mathbf{R} x y\}]\langle \lambda x \mathbf{R} x x \rangle \langle \mathbf{a} \rangle$
(10) $[\lambda y [\lambda \mathrm{P} \lambda \mathrm{Q} \forall x \{\mathrm{P} x \rightarrow \mathrm{Q} x\}]\langle \lambda x \mathbf{K} x y \rangle \langle \lambda x \mathbf{H} x y \rangle]\langle \mathbf{a} \rangle$