The Logic of P-Algebras

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1. Introduction

In this paper, I present a modal system called ∏ (Pi), characterizing it both axiomatically and algebraically, the latter being in terms of structures called π-algebras (pi-algebras). Pi-algebras are a natural generalization of Boolean algebras with operators – a generalization in which equality is replaced by congruence in the characterizing conditions. The resulting system of modal logic is "sub-Lewis", in the sense that it is properly contained in the weakest Lewis system, S1.

2. The Basic Modal System Õ – Axiomatic Characterization

System ∏ is underwritten by a sentential language $\mathcal{L}$ whose primitive vocabulary includes two truth-functional connectives [$\sim$, &] and one modal connective [$\Box$]. The remaining logical items [$\bot$, $\top$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\Diamond$, $\Rightarrow$, $\Leftrightarrow$] may then be defined in the usual manner.

System ∏ is axiomatically characterized by the following rules, whose nick-names are given to the right.[1]

(r1)	$\hookrightarrow \Box\tau$	where τ is a tautology (thesis of Sentential Logic)	[SLN]
(r2)	$\hookrightarrow \Box(\alpha\rightarrow\beta) \rightarrow. \Box\alpha \rightarrow \Box\beta$		[K]
(r3)	$\alpha \;;\; \alpha\rightarrow\beta \;\hookrightarrow\; \beta$		[MP]
(r4–)	$\{\vdash\Box\alpha\} \;\hookrightarrow\; \alpha$		[T–][2]
(r5)	$\{\vdash\alpha\Leftrightarrow\beta\} \;\hookrightarrow\; \Box\alpha\Leftrightarrow\Box\beta$		[E]

Note that $\alpha\Leftrightarrow\beta =_{df} \Box(\alpha\leftrightarrow\beta)$. Also note that, in rules (r4–) and (r5), the notation '$\{\vdash\phi\}$' means that the prior lines constitute a *proof* of ϕ.

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[1] We present axiom systems exclusively in terms of derivation rules, including zero-place rules which substitute for axiom *schemata*. The rule marker is the broken-arrow '$\hookrightarrow$'.

[2] The reason for the "minus" notation in (r4–) is that the most natural extension of ∏, called ∏+T, is obtained by replacing (r4–) [a.k.a. T–] by (r4) [a.k.a. T], which is the characteristic rule of System T.

(r4) $\hookrightarrow\Box\alpha\rightarrow\alpha$ [T]

Given the presence of T and MP in ∏+T, it is easy to show that (r4–) is admissible in ∏+T. More about ∏+T in Section 13.

3. Boolean Algebras and Pi-Algebras

The proposed semantics for System $\prod$ is based on the notion of a pi-algebra, which is a Boolean algebra $\mathcal{B}$ augmented by the following two items.

(1) a Boolean congruence relation $\approx$ on $\mathcal{B}$;
(2) a one-place function $\Box$ on $\mathcal{B}$, subject to various restrictions (Section 7).

We regard a Boolean algebra as an algebra whose primitive operations include the following.

(b1)	$\neg$	ortho-complement operation	interprets negation ($\sim$)
(b2)	$\wedge$	meet operation	interprets conjunction (&)

The remaining Boolean operations are defined in the expected manner.

(b3)	0	zero-element	interprets $\bot$
(b4)	1	unit-element	interprets $\top$
(b5)	$\vee$	join operation	interprets disjunction ($\vee$)
(b6)	$\rightarrow$	conditional operation	interprets the conditional ($\rightarrow$)
(b7)	$\leftrightarrow$	biconditional operation	interprets the biconditional ($\leftrightarrow$)

Additionally, every Boolean algebra admits a two-place *implication relation* $\leqslant$ satisfying the following conditions.

(p1) $x \leqslant y \;\Leftrightarrow\; x \vee y = y$
(p2) $x \leqslant y \;\Leftrightarrow\; x \wedge y = x$

The relation $\leqslant$ is a *partial ordering*, which is to say it satisfies the following conditions.

(p3) $x \leqslant x$ [$\leqslant$ is reflexive]
(p4) $x \leqslant y \;\&\; y \leqslant z \;.\!\Rightarrow\; x \leqslant z$ [$\leqslant$ is transitive]
(p5) $x \leqslant y \;\&\; y \leqslant x \;.\!\Rightarrow\; x = y$ [$\leqslant$ is anti-symmetric]

4. Boolean Congruences

Let $\mathcal{B}$ be a Boolean algebra, and let $\approx$ be a two-place relation on $\mathcal{B}$. Then $\approx$ is said to be a *congruence relation* on $\mathcal{B}$ if and only if it is an equivalence relation that "respects" the Boolean operations. In other words,

(c1) $x \approx x$ [$\approx$ is reflexive]
(c2) $x \approx y \Rightarrow y \approx x$ [$\approx$ is symmetric]
(c3) $x \approx y \;\&\; y \approx z \;.\!\Rightarrow\; x \approx z$ [$\approx$ is transitive]
(c4) $x \approx y \Rightarrow \neg x \approx \neg y$ [$\approx$ respects $\neg$]
(c5) $x \approx x' \;\&\; y \approx y' \;.\!\Rightarrow\; x \wedge y \approx x' \wedge y'$ [$\approx$ respects $\wedge$]

5. Pi-Orderings

Let $\mathcal{B}$ be a Boolean algebra, and let $\approx$ be a congruence on $\mathcal{B}$. Define an affiliated order-relation $\precsim$ on $\mathcal{B}$ as follows.

(d1) $x \precsim y \quad =_{df} \quad x \approx x \wedge y$

We call $\precsim$ is called a *pi-ordering* (π-ordering), where 'pi' is short for 'para-implication'. Basically, a para-implication relation "hovers" near or around the fundamental implication relation [i.e., $\leqslant$].

First, it is easy to show that $\precsim$ extends $\leqslant$, and is a quasi-ordering [i.e., reflexive and transitive relation] over $\mathcal{B}$.

(t1) $x \leqslant y \Rightarrow x \precsim y$ [$\precsim$ extends $\leqslant$]
(t2) $x \precsim x$ [$\precsim$ is reflexive]
(t3) $x \precsim y \;\&\; y \precsim z \;.\!\!\Rightarrow x \precsim z$ [$\precsim$ is transitive]

Next, every quasi-ordering gives rise to an affiliated equivalence relation. In the case of a pi-ordering $\precsim$, the affiliated equivalence relation is quite simply the original congruence relation $\approx$, as seen in the following theorem.

(t4) $x \precsim y \;\&\; y \precsim x \;.\!\!\Rightarrow x \approx y$

Other simple theorems about $\precsim$ are given as follows, along with the parallel theorems about $\leqslant$.

(t5)	$x \precsim y \Leftrightarrow x \vee y \approx y$	$x \leqslant y \Leftrightarrow x \vee y = y$
(t6)	$x \precsim y \Rightarrow \neg y \precsim \neg x$	$x \leqslant y \Rightarrow \neg y \leqslant \neg x$
(t7)	$x \precsim y \;\&\; x \precsim z \;.\!\!\Rightarrow x \precsim y \wedge z$	$x \leqslant y \;\&\; x \leqslant z \;.\!\!\Rightarrow x \leqslant y \wedge z$
(t8)	$x \precsim z \;\&\; y \precsim z \;.\!\!\Rightarrow x \vee y \precsim z$	$x \leqslant z \;\&\; y \leqslant z \;.\!\!\Rightarrow x \vee y \leqslant z$
(t9)	$x \wedge y \precsim z \Leftrightarrow x \precsim y \rightarrow z$	$x \wedge y \leqslant z \Leftrightarrow x \leqslant y \rightarrow z$
(t10)	$x \wedge (x \rightarrow y) \precsim y$	$x \wedge (x \rightarrow y) \leqslant y$
(t11)	$x \precsim y \Leftrightarrow x \rightarrow y \approx 1$	$x \leqslant y \Leftrightarrow x \rightarrow y = 1$
(t12)	$x \approx y \Leftrightarrow x \leftrightarrow y \approx 1$	$x = y \Leftrightarrow x \leftrightarrow y = 1$

6. Boolean-Filters and Pi-Filters

Let $\mathcal{B}$ be a Boolean algebra. Then a *filter* on $\mathcal{B}$ is a subset $\mathbb{F}$ of $\mathcal{B}$ satisfying the following conditions.

(f1) $1 \in \mathbb{F}$
(f2) $p \in \mathbb{F} \;\&\; p \leqslant q \;.\!\!\Rightarrow q \in \mathbb{F}$
(f3) $p \in \mathbb{F} \;\&\; q \in \mathbb{F} \;.\!\!\Rightarrow p \wedge q \in \mathbb{F}$

Let $\langle \mathcal{B}, \precsim \rangle$ be a pi-ordered Boolean algebra. Then, a *pi-filter* on $\langle \mathcal{B}, \precsim \rangle$ is a subset $\mathbb{F}$ of $\mathcal{B}$ satisfying the following conditions.

(π1) $1 \in \mathbb{F}$
(π2) $p \in \mathbb{F} \;\&\; p \precsim q \;.\!\!\Rightarrow q \in \mathbb{F}$
(π3) $p \in \mathbb{F} \;\&\; q \in \mathbb{F} \;.\!\!\Rightarrow p \wedge q \in \mathbb{F}$

Notice that, since $x \precsim y$ if $x \leqslant y$, every π-filter is automatically a filter.

7. Box-Functions

In addition to a congruence, and associated pi-ordering, a π-algebra also comes equipped with a one-place "box" function $\Box$, which interprets the necessity operator. A box-function $\Box$ is postulated to satisfy the following conditions.

(β1) $\Box(x) \approx 1 \;\Leftrightarrow\; x = 1$

(β2) $\Box(x \wedge y) \;\approx\; \Box(x) \wedge \Box(y)$

In this connection, we cite the following simple but important theorems about box-functions.

(t13) $\Box(x \to y) \wedge \Box(x) \precsim \Box(y)$
(t14) $\Box(x \to y) \precsim \Box(x) \to \Box(y)$

8. Logical Matrices based on Pi-Algebras

The method of logical matrices is well-known in logic. Given a formal sentential language $\mathcal{L}$, a *logical matrix* for $\mathcal{L}$ is a structure $\langle \mathcal{A}, \mathbb{D} \rangle$, where $\mathcal{A}$ is an algebra that is type-appropriate to $\mathcal{L}$, and $\mathbb{D}$ is a subset of $\mathcal{A}$ of "designated" elements. Where $\mathcal{M} = \langle \mathcal{A}, \mathbb{D} \rangle$ is a logical matrix, an $\mathcal{M}$*-admissible valuation* on $\mathcal{L}$ is any *homomorphism* from the algebra of formulas of $\mathcal{L}$ into the algebra $\mathcal{A}$. An admissible valuation υ is said to *satisfy* a formula ϕ [written: $\upsilon \vDash \phi$] precisely if $\upsilon(\phi) \in \mathbb{D}$. Similarly, a matrix $\mathcal{M}$ is said to *satisfy* ϕ [written: $\mathcal{M} \vDash \phi$] precisely if every $\mathcal{M}$-admissible valuation satisfies ϕ, and a class $\mathcal{K}$ of matrices is said to *satisfy* ϕ [written $\mathcal{K} \vDash \phi$] precisely if every matrix in $\mathcal{K}$ satisfies ϕ. Finally, a formula ϕ is said to be *valid* in $\mathcal{K}$ precisely if $\mathcal{K}$ satisfies ϕ.

The class $\mathcal{K}(\pi)$ *pi-matrices* is characterized as follows — a logical matrix $\mathcal{M} = \langle \mathcal{A}, \mathbb{D} \rangle$ is a pi-matrix if and only if $\mathcal{A}$ is a pi-algebra, and $\mathbb{D}$ is a π-filter on $\mathcal{A}$.

9. Lemmas About Pi-Matrices

In the present section, we cite some simple theorems about pi-matrices that are useful in proving soundness and completeness results.

(L1) Let $\mathcal{M} = \langle \langle \mathbb{B}, \approx, \Box \rangle, \mathbb{D} \rangle$ be a pi-matrix, and let υ be an $\mathcal{M}$-admissible valuation. Suppose τ is a thesis of classical SL. Then $\upsilon(\tau) = 1$.

(L2) Let $\mathcal{M} = \langle \langle \mathbb{B}, \approx, \Box \rangle, \mathbb{D} \rangle$ be a pi-matrix, and let υ be an $\mathcal{M}$-admissible valuation. Then, if ϕ is $\mathcal{M}$-valid, $\upsilon(\phi) \approx 1$. So, *a fortiori*, if ϕ is valid in $\mathcal{K}(\pi)$, then $\upsilon(\phi) \approx 1$

(L3) Let $\mathcal{M} = \langle \langle \mathbb{B}, \approx, \Box \rangle, \mathbb{D} \rangle$ be a pi-matrix, and let υ be an $\mathcal{M}$-admissible valuation. Suppose $\upsilon(\phi) \approx 1$. Then, since $\mathbb{D}$ is required to contain 1, $\upsilon(\phi) \in \mathbb{D}$, and so $\upsilon \vDash \phi$.

10. Soundness Theorem

> Every thesis of System $\prod$ is valid in the class $\mathcal{K}(\pi)$ of pi-matrices.

Proof. A standard proof by strong induction. We suppose, by way of an inductive hypothesis, that every proof of length less than m produces a valid formula, to show that a proof of length m produces a valid formula. Consider a proof of ϕ of length m. Given the formulation of System $\prod$, there are a number of ways in which the last line [i.e. ϕ] can enter such a proof, which we consider case by case.

SLN. In this case, ϕ has the form $\Box\tau$, where τ is a thesis of SL. Consider an π-admissible valuation υ; we wish to show $\upsilon\vDash\Box\tau$. Since τ is a thesis of SL, by Lemma (L1), $\upsilon(\tau) = 1$. By the homomorphism requirement[3] on υ, $\upsilon(\Box\tau) = \blacksquare(\upsilon(\tau))$, so $\upsilon(\Box\tau) = \blacksquare(\mathit{1})$. By condition ($\beta$1), $\blacksquare(1) \approx \mathit{1}$. So, $\upsilon(\Box\tau) \approx \mathit{1}$. So by Lemma (L3), $\upsilon\vDash\Box\tau$.

K. In this case, ϕ has the form $[\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)]$. Let $\upsilon(\alpha)=p$ and $\upsilon(\beta)=q$. Then by the homomorphism requirement, $\upsilon\{\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)\} = \blacksquare(p\to q)\to(\blacksquare(p)\to\blacksquare(q))$. So, in order to show that υ satisfies $[\Box(\alpha\to\beta)\to(\Box\alpha\to\Box\beta)]$, in virtue of (L3), it suffices to show that $\blacksquare(p\to q)\to[\blacksquare(p)\to\blacksquare(q)] \approx 1$. But by (t9), the latter is equivalent to $\blacksquare(p\to q) \precsim \blacksquare(p)\to\blacksquare(q)$, which is the content of (t14).

MP. In this case**,** the formula ϕ follows by *modus ponens* from earlier formulas — $\alpha\to\phi$ and α**.** Consider an admissible valuation υ; we wish to show that $\upsilon\vDash\phi$, which is to say $\upsilon(\phi)\in\mathbb{D}$**.** By the inductive hypothesis, α and $\alpha\to\phi$ are both valid; so, $\upsilon(\alpha)\in\mathbb{D}$, and $\upsilon(\alpha\to\phi)\in\mathbb{D}$**.** Let $\upsilon(\alpha)=p$, and $\upsilon(\phi)=q$. Then by the homomorphism requirement, $\upsilon(\alpha\to\phi) = p\to q$. So $p\in\mathbb{D}$, and $p\to q\in\mathbb{D}$. Since $\mathbb{D}$ is a filter, it is closed under meet ($\wedge$), so $p\wedge(p\to q) \in \mathbb{D}$. But by (t10)(b), $p\wedge(p\to q) \leqslant q$. Therefore, since $\mathbb{D}$ is a filter, it is closed under implication ($\leqslant$). Therefore, $q\in\mathbb{D}$, so $\upsilon(\phi)\in\mathbb{D}$.

T– In this case, the formula ϕ follows from an *earlier proof* of $\Box\phi$. To show that ϕ is valid, consider an arbitrary valuation υ, to show that $\upsilon\vDash\phi$. Let $\upsilon(\phi)=p$. Then by the homomorphism requirement, $\upsilon(\Box\phi) = \blacksquare(p)$. By the inductive hypothesis, $\Box\phi$ is valid, so by Lemma (L2), $\upsilon(\Box\phi)\approx\mathit{1}$, so $\blacksquare(p)\approx\mathit{1}$. So by the box-restriction (β1), $p=\mathit{1}$, so $p\in\mathbb{D}$, so $\upsilon(\phi)\in\mathbb{D}$, so $\upsilon\vDash\phi$.

E. In this case, the formula ϕ has the form $\Box\alpha\Leftrightarrow\Box\beta$ [$=_{df}$ $\Box(\Box\alpha\leftrightarrow\Box\beta)$], and there is an *earlier proof* of $\alpha\Leftrightarrow\beta$ [$=_{df}$ $\Box(\alpha\leftrightarrow\beta)$]. To show $\Box(\Box\alpha\leftrightarrow\Box\beta)$ is valid, consider an arbitrary valuation υ; let $\upsilon(\alpha)=p$, and $\upsilon(\beta)=q$. By the homomorphism requirement, $\upsilon(\Box(\alpha\leftrightarrow\beta)) = \blacksquare(p\leftrightarrow q)$. By the inductive hypothesis, $\Box(\alpha\leftrightarrow\beta)$ is valid, so by Lemma (L2), $\upsilon(\Box(\alpha\leftrightarrow\beta)) \approx 1$, so $\blacksquare(p\leftrightarrow q) \approx 1$. So by the box-restriction (β1), $p\leftrightarrow q=1$, so by a BA-theorem, $p=q$. From this it immediately follows that $\blacksquare(p) = \blacksquare(q)$, and so by (t12)(b), $\blacksquare(p)\leftrightarrow\blacksquare(q) = 1$. So by ($\beta$1), $\blacksquare(\blacksquare(p)\leftrightarrow\blacksquare(q)) \approx 1$, and therefore by Lemma (L3), υ satisfies $\blacksquare(\blacksquare(p)\leftrightarrow\blacksquare(q))$.

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[3] The "homomorphism requirement" is the requirement that an admissible valuation is a homomorphism from the algebra of formulas of $\mathcal{L}$ into the algebra $\mathcal{A}$ of semantic values. In our case, this means the following.

(1) $\upsilon(\Box\alpha) = \blacksquare\{\upsilon(\alpha)\}$

(2) $\upsilon(\sim\alpha) = \neg\{\upsilon(\alpha)\}$

(3) $\upsilon(\alpha\&\beta) = \upsilon(\alpha) \wedge \upsilon(\beta)$

11. Lemmas About Axiom System Õ

Before proving the corresponding completeness theorem, we state some simple but important lemmas about the System ∏.

(L4) $\vdash\alpha$ & $\vdash\alpha\rightarrow\beta$.⇒ $\vdash\beta$

(L5) $\vdash\Box\tau$, for any SL-thesis τ

(L6) $\vdash\Box\alpha \Rightarrow \vdash\alpha$

(L7) $\vdash\beta \Rightarrow \vdash\alpha\rightarrow\beta$

(L8) $\vdash\Box(\alpha\rightarrow\beta) \Rightarrow \vdash\Box\alpha\rightarrow\Box\beta$

(L9) $\vdash\Box(\alpha\rightarrow\beta)$ & $\vdash\Box\alpha$.⇒ $\vdash\Box\beta$

(L10) $\vdash\alpha\leftrightarrow\beta$ ⇒. $\vdash\alpha \Leftrightarrow \vdash\beta$

(L11) $\vdash\alpha\Leftrightarrow\alpha$

(L12) $\vdash\alpha\Leftrightarrow\beta \Rightarrow \vdash\beta\Leftrightarrow\alpha$

(L13) $\vdash\alpha\Leftrightarrow\beta$ & $\vdash\beta\Leftrightarrow\gamma$.⇒ $\vdash\alpha\Leftrightarrow\gamma$

(L14) $\vdash\alpha\Leftrightarrow\beta \Rightarrow \vdash{\sim}\alpha \Leftrightarrow {\sim}\beta$

(L15) $\vdash\alpha \Leftrightarrow\alpha'$ & $\vdash\beta\Leftrightarrow\beta'$.⇒ $\vdash(\alpha\&\beta) \Leftrightarrow (\alpha'\&\beta')$

(L16) $\vdash\alpha\Leftrightarrow\beta \Rightarrow \vdash\Box\alpha \Leftrightarrow \Box\beta$

(L17) $\vdash\Box(\alpha\&\beta)\leftrightarrow(\Box\alpha \,\&\, \Box\beta)$

(L18) $\vdash\Box(\alpha\leftrightarrow\beta)$ ⇔. $\vdash\Box(\alpha\rightarrow\beta)$ & $\vdash\Box(\beta\rightarrow\alpha)$

(L19) $\vdash\alpha\leftrightarrow\beta$ ⇔. $\vdash\alpha\rightarrow\beta$ & $\vdash\beta\rightarrow\alpha$

(L20) $\vdash\alpha$ & $\vdash\beta$.⇔ $\vdash\alpha\&\beta$

12. Completeness Theorem

> Every formula that is valid in the class $\mathcal{K}(\pi)$ of π-matrices is a thesis of System ∏.

Proof. Given a formula ϕ that is *not* a thesis of system ∏, we wish to show that at least one π-admissible valuation refutes ϕ. Indeed, we can define a *single* π-admissible valuation that refutes *every* non-thesis. Define the π-matrix and associated valuation as follows.

First, define the equivalence relation ≡, and the affiliated equivalence classes, as follows.

(d2) $\alpha \equiv \beta =_{df} \vdash\alpha\Leftrightarrow\beta$ [i.e., $\vdash\Box(\alpha\leftrightarrow\beta)$]

(d3) $[\![\alpha]\!] =_{df} \{\beta : \alpha \equiv \beta\}$

Define matrix $\mathcal{M} = \langle\langle\langle \mathcal{A}, \neg, \wedge, \ldots\rangle \approx, \blacksquare\rangle, \mathbb{D}\rangle$ as follows.

(d4) $\mathcal{A}$ consists of equivalence classes of formulas under the ≡ relation;
i.e., $\mathcal{A} = \{[\![\alpha]\!] : \alpha \in \mathcal{L}\}$

(d5) $1 =_{df} [\![\top]\!]$ [where $\top =_{df} {\sim}(P\&{\sim}P)$]

(d6) $\blacksquare[\![\alpha]\!] \;=_{df}\; [\![\Box\alpha]\!]$

(d7) $\neg[\![\alpha]\!] \;=_{df}\; [\![\sim\alpha]\!]$

(d8) $[\![\alpha]\!] \wedge [\![\beta]\!] \;=_{df}\; [\![\alpha \,\&\, \beta]\!]$

(d9) $[\![\alpha]\!] \approx [\![\beta]\!] \;=_{df}\; \vdash\alpha\leftrightarrow\beta$

(d10) $\mathbb{D} \;=_{df}\; \{[\![\alpha]\!] : \vdash\alpha\}$

(d11) $\upsilon(\alpha) \;=_{df}\; [\![\alpha]\!]$

We wish to show that υ is a π-admissible valuation that refutes every non-thesis of Π. We divide the proof into several steps, as follows.

(1) $\langle\mathcal{A}, \neg, \wedge\rangle$ is a Boolean algebra;

(2) $\langle\, \langle\mathcal{A}, \neg, \wedge\rangle \approx, \blacksquare \,\rangle$ is a Pi-algebra;

(3) $\mathbb{D}$ is a π-filter on $\langle\, \langle\mathcal{A}, \neg, \wedge\rangle \approx, \blacksquare \,\rangle$;

(4) υ is a π-admissible valuation;

(5) υ refutes ϕ, if ϕ is not a thesis of Π.

Proof.

(1) $\langle\mathcal{A}, \neg, \wedge\rangle$ is a Boolean algebra;

This requires proving that the operations $\neg$ and $\wedge$ are well-defined, which amounts to proving that the $\equiv$ is an equivalence relation (a-c) that respects both negation (d) and conjunction (e).

(a) $\alpha \equiv \alpha$
i.e., $\vdash\alpha\Leftrightarrow\alpha$ see (L11)

(b) $\alpha \equiv \beta \;\Rightarrow\; \beta \equiv \alpha$
i.e., $\vdash\alpha\Leftrightarrow\beta \;\Rightarrow\; \vdash\beta\Leftrightarrow\alpha$ see (L12)

(c) $\alpha \equiv \beta \;\&\; \beta \equiv \gamma \;.\!\Rightarrow\; \alpha \equiv \gamma$
i.e., $\vdash\alpha\Leftrightarrow\beta \;\&\; \vdash\beta\Leftrightarrow\gamma \;.\!\Rightarrow\; \vdash\alpha\Leftrightarrow\gamma$ see (L13)

(d) $\alpha \equiv \beta \;\Rightarrow\; \sim\alpha \equiv \sim\beta$
i.e., $\vdash\alpha\Leftrightarrow\beta \;\Rightarrow\; \vdash\sim\alpha\Leftrightarrow\sim\beta$ see (L14)

(e) $\alpha \equiv \alpha' \;\&\; \beta \equiv \beta' \;.\!\Rightarrow\; (\alpha\&\beta) \equiv (\alpha'\&\beta')$
i.e., $\vdash\alpha\Leftrightarrow\alpha' \;\&\; \vdash\beta\Leftrightarrow\beta' \;.\!\Rightarrow\; \vdash\alpha\&\beta \;.\!\Leftrightarrow\!.\; \alpha'\&\beta'$ see (L15)

This means that the operations $\neg$ and $\wedge$ are well-defined. That the operations behave in a Boolean manner — e.g., $[\![\alpha]\!]\wedge[\![\beta]\!] = [\![\beta]\!]\wedge[\![\alpha]\!]$ — is shown routinely.

(2) $\langle\, \langle\mathcal{A}, \neg, \wedge \,\rangle \approx, \blacksquare\rangle$ is a Pi-algebra

This requires proving that $\approx$ and $\blacksquare$ are well-defined, which amounts to showing the following.

(a) $\alpha \equiv \alpha' \;\&\; \beta \equiv \beta' \;.\!\Rightarrow\!.\; \alpha \approx \beta \;\Leftrightarrow\; \alpha' \approx \beta'$
i.e., $\vdash\alpha\Leftrightarrow\alpha' \;\&\; \vdash\beta\Leftrightarrow\beta' \;.\!\Rightarrow\!.\; \vdash\alpha\leftrightarrow\beta \;\Leftrightarrow\; \vdash\alpha'\leftrightarrow\beta'$

Suppose $\vdash\Box(\alpha\leftrightarrow\alpha')$, and $\vdash\Box(\beta\leftrightarrow\beta')$. Then by (L6), $\vdash\alpha\leftrightarrow\alpha'$, and $\vdash\beta\leftrightarrow\beta'$. Now by SL, $\vdash(\alpha\leftrightarrow\alpha')\to[\beta\leftrightarrow\beta'.\to:\alpha\leftrightarrow\beta.\leftrightarrow.\alpha'\leftrightarrow\beta']$, so by two applications of (L4), $\vdash\alpha\leftrightarrow\beta.\leftrightarrow.\alpha'\leftrightarrow\beta'$, so by (L10), $\vdash\alpha\leftrightarrow\beta \Leftrightarrow \vdash\alpha'\leftrightarrow\beta'$.

(b) $\alpha\equiv\beta \Rightarrow \Box\alpha\equiv\Box\beta$
i.e., $\vdash\alpha\Leftrightarrow\beta \Rightarrow \vdash\Box\alpha \Leftrightarrow \Box\beta$ see (L16)

We must also show that $\approx$ is a Boolean congruence, which requires showing the following.

(a) $[\![\alpha]\!]\approx[\![\beta]\!] \Rightarrow \neg[\![\alpha]\!]\approx\neg[\![\beta]\!]$
i.e., $\vdash\alpha\leftrightarrow\beta \Rightarrow \vdash{\sim}\alpha\leftrightarrow{\sim}\beta$

(b) $[\![\alpha]\!]\approx[\![\alpha']\!]$ & $[\![\beta]\!]\approx[\![\beta']\!]$ $.\Rightarrow$ $[\![\alpha]\!]\wedge[\![\beta]\!]\approx[\![\alpha']\!]\wedge[\![\beta']\!]$
i.e., $\vdash\alpha\leftrightarrow\alpha'$ & $\vdash\beta\leftrightarrow\beta'$ $.\Rightarrow$ $\vdash(\alpha\&\beta)\leftrightarrow(\alpha'\&\beta')$

These are both standard theorems about any system that contains SL, so we omit the proofs here.

Finally, we must show that $\blacksquare$ is a box-operator, which requires showing that it satisfies restrictions (β1) and (β2).

(β1) $\blacksquare[\![\alpha]\!]\approx 1 \Leftrightarrow [\![\alpha]\!]=1$
i.e., $\blacksquare[\![\alpha]\!]\approx[\![\top]\!] \Leftrightarrow [\![\alpha]\!]=[\![\top]\!]$
i.e., $\vdash\Box\alpha\leftrightarrow\top \Leftrightarrow \vdash\alpha\Leftrightarrow\top$
i.e., $\vdash\Box\alpha\leftrightarrow\top \Leftrightarrow \vdash\Box(\alpha\leftrightarrow\top)$

[$\Rightarrow$] Suppose $\vdash\Box\alpha\leftrightarrow\top$. In virtue of its definition, $\top$ is an SL-thesis, so by (L5) and (L6), $\vdash\top$, so by (L10), $\vdash\Box\alpha$. Also, $\alpha\to(\top\to\alpha)$ is an SL-thesis, so by (L5), $\vdash\Box(\alpha\to(\top\to\alpha))$, so by (L9), $\vdash\Box(\top\to\alpha)$. Similarly, $\vdash\Box(\top\to(\alpha\to\top))$, so by (L9), $\vdash\Box(\alpha\to\top)$, so by (L17), $\vdash\Box(\alpha\leftrightarrow\top)$.
[$\Leftarrow$] Suppose $\vdash\Box(\alpha\leftrightarrow\top)$. Then by (L17), $\vdash\Box(\top\to\alpha)$, so by (L8), $\vdash\Box\top\to\Box\alpha$. $\top$ is an SL-thesis, so by (L5), $\vdash\Box\top$, so by (L9), $\vdash\Box\alpha$, so by (L7), $\vdash\top\to\Box\alpha$. As in [$\leftarrow$], $\vdash\top$, so by (L7), $\vdash\Box\alpha\to\top$, so by (L19), $\vdash\Box\alpha\leftrightarrow\top$.

(β2) $\blacksquare([\![\alpha]\!]\wedge[\![\beta]\!]) \approx \blacksquare[\![\alpha]\!]\wedge\blacksquare[\![\beta]\!]$
i.e., $[\![\Box(\alpha\&\beta)]\!]\approx[\![\Box\alpha\ \&\ \Box\beta]\!]$
i.e., $\vdash\Box(\alpha\&\beta)\leftrightarrow(\Box\alpha\ \&\ \Box\beta)$ see (L17)

(3) $\mathbb{D}$ is a π-filter on $\langle\langle\mathcal{A},\neg,\wedge\rangle\approx,\blacksquare\rangle$

(f0) $\mathbb{D}$ is well-defined
(f1) $1\in\mathbb{D}$
(f2) $x\in\mathbb{D}$ & $y\in\mathbb{D}$ $.\Rightarrow$ $x\wedge y\in\mathbb{D}$
(f3) $x\in\mathbb{D}$ & $x\precsim y$ $.\Rightarrow$ $y\in\mathbb{D}$

(f0) Since $\mathbb{D}=_{df}\{[\![\alpha]\!]:\vdash\alpha\}$, this amounts to proving that
$\alpha\equiv\beta \Rightarrow. \vdash\alpha \Leftrightarrow \vdash\beta$
i.e., $\vdash\alpha\Leftrightarrow\beta \Rightarrow. \vdash\alpha \Leftrightarrow \vdash\beta$
This is an immediate consequence of (L6) and (L10).

(f1) Since $1=_{df}[\![\top]\!]$, and $\top$ is an SL-thesis; this follows from (L5)and (L6).

(f2) $[\![\alpha]\!]\in\mathbb{D}$ & $[\![\beta]\!]\in\mathbb{D}$ $.\Rightarrow$ $[\![\alpha]\!]\wedge[\![\beta]\!]\in\mathbb{D}$
i.e., $\vdash\alpha$ & $\vdash\beta$ $.\Rightarrow$ $\vdash\alpha\&\beta$. see (L20).

(f3) $[\![\alpha]\!] \in \mathbb{D}$ & $[\![\alpha]\!] \precsim [\![\beta]\!]$.$\Rightarrow$ $[\![\beta]\!] \in \mathbb{D}$;
i.e., $\vdash\alpha$ & $\vdash\alpha\rightarrow\beta$.$\Rightarrow$ $\vdash\beta$ see (L4).

(4) υ is π-admissible;

Clearly, υ maps the formulas of $\mathscr{L}$ into elements of the matrix. The only remaining question is whether υ satisfies the homomorphism requirement. This comes down to three cases.

(h1) $\upsilon(\sim\alpha) = \neg\{\upsilon(\alpha)\}$
i.e., $[\![\sim\alpha]\!] = \neg[\![\alpha]\!]$
This is immediately true in virtue of the definition of $\neg$.

(h2) $\upsilon(\Box\alpha) = \blacksquare\{\upsilon(\alpha)\}$
i.e., $[\![\Box\alpha]\!] = \blacksquare[\![\alpha]\!]$
This is immediately true in virtue of the definition of $\blacksquare$.

(h2) $\upsilon(\alpha\&\beta) = \upsilon(\alpha)\wedge\upsilon(\beta)$
i.e., $[\![\alpha\&\beta]\!] = [\![\alpha]\!]\wedge[\![\beta]\!]$
This is immediately true in virtue of the definition of $\wedge$.

(5) υ refutes ϕ, if ϕ is not a thesis of Π.

To say that υ refutes ϕ is just to say that υ does not satisfy ϕ, which amounts to saying that $\upsilon(\phi)\notin\mathbb{D}$. By definition, $\mathbb{D} = \{[\![\alpha]\!] : \vdash\alpha\}$, and $\upsilon(\phi) = [\![\phi]\!]$, so the question whether $\upsilon(\phi)\in\mathbb{D}$ is just the question whether ϕ is a thesis of Π [i.e., $\vdash\phi$]. So if ϕ is *not* a thesis of Π [i.e., $\nvdash\phi$], then $\upsilon(\phi)\notin\mathbb{D}$, and so υ refutes ϕ.

13. System Õ Is Properly Contained in Lewis' System S1

In the current section, we show that

(a) every thesis of $\prod$ is a thesis of S1;
(b) not every thesis of S1 is a thesis of $\prod$.

In order to show that every thesis of $\prod$ is a thesis of S1, we rely on a result by Lemmon[4] that a system he calls S0.9 is contained in S1. System S0.9 may be axiomatically characterized by the following rules, labeled in direct correspondence with System $\prod$.

(r1)	$\hookrightarrow \Box\tau$	where τ is a tautology (thesis of Sentential Logic)	[SLN]
(r2)	$\hookrightarrow \Box(\alpha\rightarrow\beta) \rightarrow. \Box\alpha \rightarrow \Box\beta$		[K]
(r2+)	$\hookrightarrow \Box\{\Box(\alpha\rightarrow\beta) \rightarrow. \Box\alpha \rightarrow \Box\beta\}$		[K+]
(r3)	α ; $\alpha\rightarrow\beta \hookrightarrow \beta$		[MP]
(r4)	$\hookrightarrow \Box\alpha\rightarrow\alpha$		[T]
(r4+)	$\hookrightarrow \Box\{\Box\alpha\rightarrow\alpha\}$		[T+]
(r5)	$\{\vdash\alpha\Leftrightarrow\beta\} \hookrightarrow \Box\alpha \Leftrightarrow \Box\beta$		[E]

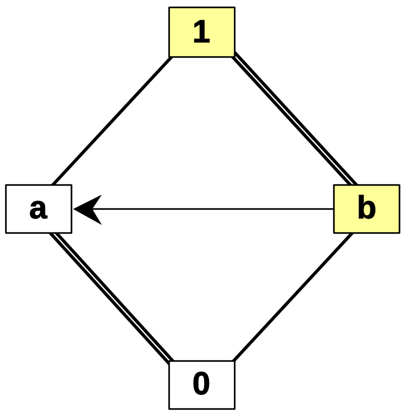
Notice that every schema of the enlarged system Π+T (see footnote 2) is included in the above list. So system Π+T, and hence Π, is contained in S0.9, and hence S1.

[4] *Journal of Symbolic Logic*, 22(1957), 176-186.

To show that S0.9, and hence S1, is not included in Π, we note that neither (r2+) nor (r4+) is a valid schema of System Π. To see this, consider the following instances.

$\Box\{\Box P\rightarrow Q\}$
$\Box\{\Box(P\rightarrow Q)\rightarrow(\Box P\rightarrow\Box Q)\}$

Consider a π-matrix based on Boolean algebra $\mathcal{B}4$, whose Hasse diagram is given below. In this particular case, the congruence is depicted by the double-lines, and the non-trivial component of the $\boldsymbol{\Box}$-function is depicted by the arrow [i.e., $\boldsymbol{\Box}(b)=a$, and $\boldsymbol{\Box}(x)=x$ if $x\neq b$]. Also, $\mathbb{D}$ = $\{1,b\}$.



Consider the following valuation: $\upsilon(P) = b$; $\upsilon(Q) = 0$

Then,
$\upsilon(\Box(\Box P\rightarrow P)) = \boldsymbol{\Box}(\boldsymbol{\Box}(b)\rightarrow b)$
$\boldsymbol{\Box}\{\boldsymbol{\Box}(b\rightarrow 0)\rightarrow(\boldsymbol{\Box}(b)\rightarrow\boldsymbol{\Box}(0))\}$
$\boldsymbol{\Box}(b) = a$
$\boldsymbol{\Box}(b)\rightarrow b = a\rightarrow b = b$
$\boldsymbol{\Box}(\boldsymbol{\Box}(b)\rightarrow b) = \boldsymbol{\Box}(b) = a$
But $a \notin \mathbb{D}$
so,
$\upsilon \not\models \Box(\Box P\rightarrow P)$

Also
$\upsilon(\Box\{\Box(P\rightarrow Q)\rightarrow(\Box P\rightarrow\Box Q)\}) =$
$\boldsymbol{\Box}\{\boldsymbol{\Box}(b\rightarrow 0) \rightarrow (\boldsymbol{\Box}(b)\rightarrow\boldsymbol{\Box}(0))\}$
$b\rightarrow 0 = a$;
$\boldsymbol{\Box}(b\rightarrow 0) = \boldsymbol{\Box}a = a$
$\boldsymbol{\Box}(b) = a$
$\boldsymbol{\Box}(0) = 0$
$\boldsymbol{\Box}(b)\rightarrow\boldsymbol{\Box}(0) = a\rightarrow 0 = b$
$\boldsymbol{\Box}(b\rightarrow 0)\rightarrow(\boldsymbol{\Box}(b)\rightarrow\boldsymbol{\Box}(0)) = a\rightarrow b = b$
$\boldsymbol{\Box}\{\boldsymbol{\Box}(b\rightarrow 0)\rightarrow(\boldsymbol{\Box}(b)\rightarrow\boldsymbol{\Box}(0))\} = \boldsymbol{\Box}(b) = a$
But $a \notin \mathbb{D}$, so:
$\upsilon \not\models \Box\{\Box(P\rightarrow Q)\rightarrow(\Box P\rightarrow\Box Q)\}$