$\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\}$, since it was new when we introduced it. So there is no reason it must be used rather than the variable.

6. Hence, $\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\} \vdash \delta_m[\zeta] \Rightarrow \beta$.

7. By Gen, $\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\} \vdash (\forall\zeta)(\delta_m[\zeta] \Rightarrow \beta)$.

8. Because ζ does not occur free in the proof, and β is α_n, β does not contain ζ free. Hence, by (New-DR), $\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\} \vdash (\exists\zeta)\delta_m[\zeta] \Rightarrow \beta$.

9. Because $\delta_m[\kappa_m]$ was arrived at in the original pseudo-derivation by EI on some wff of the form $(\exists\xi_m)\delta_m[\xi_m]$, which either *is* $(\exists\zeta)\delta_m[\zeta]$, or can be used to get it, it must be that $\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\} \vdash (\exists\zeta)\delta_m[\zeta]$. Thus, by MP, $\Gamma \cup \{\delta_1[\kappa_1], \ldots, \delta_{m\text{-}1}[\kappa_{m\text{-}1}]\} \vdash \beta$.

10. By the same procedure described in steps 4-9, we can eliminate $\delta_{m\text{-}1}[\kappa_{m\text{-}1}]$ from the premise set, and so on, until we have eliminated everything except the members of Γ.
Hence, $\Gamma \vdash \beta$.

This proof shows us that we don't need Existential Instantiation, very much like we don't need Conditional Proof. However, because we have this metatheoretic result, we now know that whenever we are able to carry out a pseudo-proof making use of the rule, we could transform it into a proof that does not make use of the rule. Hence, it is innocuous to pretend as if we do have such a rule, even though strictly speaking we do not. Hence, from here on out we allow ourselves to make use of "Rule C" or "EI" in our proofs, even though strictly speaking there is no such rule. We must be careful, however, to obey the restrictions for what counts as a "pseudo-derivation", as defined in the previous page. I like to mark the pseudo-steps with "$\vdash$*" rather than "$\vdash$"; you can stop using the "*" when the dummy constant no longer appears.

VII. Metatheoretic Results for System PF

Soundness: For all wffs α, if $\vdash \alpha$ then $\vDash \alpha$.

Proof: Every instance of the axiom schemata is logically valid. (This can be verified using semantic trees.) *MP* and *Gen* preserve logical validity. (In the case of *Gen*, note that a wff is logically valid iff it is satisfied by all sequences in all models. If an open wff is satisfied by all sequences in a model, then the corresponding wff with one of the variables bound with an initial quantifier will also be satisfied by all sequences in that model.) If $\vdash \alpha$, then α is derivable from the axioms by some finite number of steps of MP and Gen, each preserving validity, and hence, $\vDash \alpha$.

Consistency: There is no wff α such that both $\vdash \alpha$ and $\vdash \neg\alpha$.

Proof: Suppose for *reductio ad absurdum* that there is a wff α such that $\vdash \alpha$ and $\vdash \neg\alpha$. By soundness, $\vDash \alpha$ and $\vDash \neg\alpha$. In other words, every sequence in every model satisfies both α and $\neg\alpha$. But a sequence satisfies $\neg\alpha$ iff it does not satisfy α, so any arbitrary sequence will both satisfy and not satisfy α, which is absurd.

- Ultimately, we also want to prove the completeness of PF. We'll get there, but we first need to prove a number of lemmas.

Denumerability of the set of wffs: The set of wffs of the language of predicate logic is denumerable, i.e., we can generate a one–one correspondence between the set of natural numbers the set of wffs.

Proof: All wffs are built up of the simple signs: '(', ',', ')', '$\Rightarrow$', '$\neg$', '$\forall$', as well as the individual constants, variables, predicate letters and function letters.

A. We define a function *g* that associates each simple sign with a different natural number.

1. Let $g(\text{'('}) = 3$, $g(\text{')'}) = 5$, $g(\text{','}) = 7$, $g(\text{'}\neg\text{'}) = 9$, $g(\text{'}\Rightarrow\text{'}) = 11$, and $g(\text{'}\forall\text{'}) = 13$.
2. If κ is a constant, and n is the number of its subscript (if κ has no subscript, then $n = 0$), then depending on which letter of the alphabet is used, let k be either 1, 2, 3, 4 or 5 (1 for '*a*', 2 for '*b*', etc.), and let $g(\kappa) = 7 + 8(5n + k)$.
3. If ξ is a variable, and n is the number of its subscript (if ξ has no subscript, then $n = 0$), then

depending on which letter of the alphabet is used, let k be either 1, 2, or 3 (1 for ‘x’, 2 for ‘y’ and 3 for ‘z’), and let $g(\xi) = 13 + 8(3n + k)$.

4. If ϕ is a function letter, and n is the number of its subscript (if ϕ has no subscript, then $n = 0$) and m is the number of its superscript, then depending of which letter of the alphabet is used (‘f’ through ‘l’), let k be one of 1 through 7, and let $g(\phi) = 1 + 8(2^m3^{(7n + k)})$.

5. If π is a predicate letter, and n is the number of its subscript (if π has no subscript, then $n = 0$) and m is the number of its superscript, then depending of which letter of the alphabet is used (‘A’ through ‘T’), let k be one of 1 through 20, and let $g(\pi) = 3 + 8(2^m3^{(20n + k)})$.

B. We can now define the value of *g* for formulae in virtue of its value for simple signs.

6. Let (p_0, p_1, p_3, p_4, …) be the sequence of prime integers in order starting with 2. (There is no greatest prime.) Hence $p_0 = 2$, $p_1 = 3$, $p_3 = 5$ and so on.

7. Let $u_0u_1u_2\ldots u_r$ be some string of signs from the syntax of predicate logic. It might be something ill-formed like “$\Rightarrow)x\forall a_{12}($” or it might be a well-formed formula like “$(\forall x_1)(F(x_1) \Rightarrow F(x_1))$”. Hence, u_0 is the first sign in the string, u_1 is the second sign, and so on. For all such strings, let $g(u_0u_1u_2\ldots u_r) = p_0^{g(u_0)}p_1^{g(u_1)}p_2^{g(u_2)}\ldots p_r^{g(u_r)}$.

9. For a given expression α, the number $g(\alpha)$ is called the ***Gödel number*** of α. Notice that because the Gödel numbers of the different simple signs are all different, so are the Gödel numbers of strings of signs, since for different strings, these numbers will have different prime factorizations.

10. Let N – {0} be the set of natural numbers greater than zero, C the subset of natural numbers that are Gödel numbers of wffs, and W the set of all wffs. Consider now the function $w(x)$ from N – {0} onto C, whose value for x as argument is the x^{th} smallest natural number that is the Gödel number of a wff of predicate logic. Consider also the function $s(x)$ from C onto W, whose value for any Gödel number of a wff is that wff. Then the function $s(w(n + 1))$, is a 1–1 correspondence between the set of natural numbers and the set of wffs.

Obvious corollary: The set of *closed* wffs is also denumerable

∗∗ We’re inching closer to completeness. Before moving on, I want to make note of some differences between my proof of completeness and Mendelson’s. Mendelson prefers to speak of different *first-order theories*. Remember than a first-order theory is an axiomatic system gotten by adding additional axioms to the axioms of PF. Really, talking about what theorems are provable in a given system K, where the additional axioms of K are the members of a set Γ is equivalent to speaking about what is provable in the barebones system PF beginning with Γ as a set of premises, since clearly:

$$\vdash_{\text{K}} \alpha \text{ iff } \Gamma \vdash_{\text{PF}} \alpha$$

It’s really a matter of taste whether we give the proof as being about different theories or as being about different premise sets. I prefer to speak about premise sets, since that we don’t have to deal with any sense of “$\vdash$” other than “$\vdash_{\text{PF}}$”. But the differences are trivial.

Before moving on, let us introduce some new metalinguistic definitions:

Definition: A set of wffs Γ is said to be ***consistent*** iff there is no wff β such that both $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Otherwise, Γ is ***inconsistent***.

Definition: A set of wffs Γ is said to be ***maximal*** iff for every closed wff β, either $\beta \in \Gamma$ or $\neg\beta \in \Gamma$.

Definition: A set of wffs Γ is said to be ***universal*** iff for every wff $\beta[\xi]$ that contains at most ξ free, if it is the case for all closed terms τ that $\beta[\tau] \in \Gamma$, then $(\forall\xi)\beta[\xi] \in \Gamma$.

We now move on to our next important Lemma on the way to completeness.

Lindenbaum Extension Lemma: If Γ is a consistent set of closed wffs, then there is a set of closed wffs Δ such that: (a) $\Gamma \subseteq \Delta$, (b) Δ is consistent, (c) Δ is maximal, and (d) Δ is universal.

Proof:

1. Assume that Γ is a consistent set of closed wffs.

2. For convenience, we assume that none of the constants '*e*', 'e_1', 'e_2', 'e_3', …, etc., occur anywhere in the wffs in Γ.[†]

3. By the denumerability of the set of closed wffs of the language, we can arrange them in an infinite sequence:

$\alpha_1, \alpha_2, \alpha_3, \ldots$, etc.

Making use of this sequence, let us recursively define an infinite sequence of *sets* of wffs:

$\Gamma_0, \Gamma_1, \Gamma_2, \ldots$, etc.

As follows:

a) Let $\Gamma_0 = \Gamma$

b) We define Γ_{n+1} in terms of Γ_n in one of the following three ways:

(i) if $\Gamma_n \cup \{\alpha_{n+1}\}$ is consistent, then let $\Gamma_{n+1} = \Gamma_n \cup \{\alpha_{n+1}\}$;

(ii) if $\Gamma_n \cup \{\alpha_{n+1}\}$ is inconsistent and α_{n+1} does not take the form $(\forall\xi)\,\beta[\xi]$, then let $\Gamma_{n+1} = \Gamma_n \cup \{\neg\alpha_{n+1}\}$;

(iii) if $\Gamma_n \cup \{\alpha_{n+1}\}$ is inconsistent and α_{n+1} does take the form $(\forall\xi)\,\beta[\xi]$, then let $\Gamma_{n+1} = \Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\}$, where '$e_x$' is the first member of the sequence '*e*', 'e_1', 'e_2', 'e_3', …, etc., that does not occur in Γ_n.

4. Let Δ be the *union* of all of the members of the Γ-sequence (i.e., $\Gamma_0 \cup \Gamma_1 \cup \Gamma_2 \cup \ldots$ etc.)

5. Obviously, $\Gamma \subseteq \Delta$. This establishes part (a) of the consequent of the Lemma.

6. Every member of the Γ-sequence is consistent. We prove this by mathematical induction.

a) **Base step:** Γ_0 is Γ, and it is consistent *ex hypothesi.*

b) **Induction step:** Suppose Γ_n is consistent. It follows that Γ_{n+1} is consistent by a proof by cases:

Case (i): $\Gamma_{n+1} = \Gamma_n \cup \{\alpha_{n+1}\}$ and $\Gamma_n \cup \{\alpha_{n+1}\}$ is consistent, so Γ_{n+1} is consistent.

Case (ii): $\Gamma_{n+1} = \Gamma_n \cup \{\neg\alpha_{n+1}\}$, and $\Gamma_n \cup \{\alpha_{n+1}\}$ is inconsistent.

- Hence there is some β such that both $\Gamma_n \cup \{\alpha_{n+1}\} \vdash \beta$ and $\Gamma_n \cup \{\alpha_{n+1}\} \vdash \neg\beta$.
- By SL, $\Gamma_n \cup \{\alpha_{n+1}\} \vdash \beta \wedge \neg\beta$.
- Because, α_{n+1} is closed, it follow by DT, that $\Gamma_n \vdash \alpha_{n+1} \Rightarrow (\beta \wedge \neg\beta)$
- But $\vdash \neg(\beta \wedge \neg\beta)$ by SL.
- By MT, $\Gamma_n \vdash \neg\alpha_{n+1}$
- Suppose for *reductio* that Γ_{n+1} is inconsistent.
- So there is some δ such that $\Gamma_n \cup \{\neg\alpha_{n+1}\} \vdash \delta$ and $\Gamma_n \cup \{\neg\alpha_{n+1}\} \vdash \neg\delta$.
- By parallel reasoning as above, by SL and the deduction theorem, we also have $\Gamma_n \vdash \alpha_{n+1}$.
- So Γ_n is inconsistent.
- This contradicts the inductive hypothesis.
- Hence Γ_{n+1} is consistent.

Case (iii): $\Gamma_{n+1} = \Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\}$, $\Gamma_n \cup \{\alpha_{n+1}\}$ is inconsistent and α_{n+1} takes the form $(\forall\xi)\,\beta[\xi]$.

- By the same reasoning as in case (ii), $\Gamma_n \vdash \neg\alpha_{n+1}$.
- Suppose for *reductio* that Γ_{n+1} is inconsistent.
- So there is some δ such that $\Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\} \vdash \delta$ and $\Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\} \vdash \neg\delta$.
- By SL, $\Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\} \vdash \delta \wedge \neg\delta$.
- By DT, $\Gamma_n \cup \{\neg\beta[e_x]\} \vdash \neg\alpha_{n+1} \Rightarrow (\delta \wedge \neg\delta)$.
- By MP, $\Gamma_n \cup \{\neg\beta[e_x]\} \vdash \delta \wedge \neg\delta$
- Because α_{n+1} is closed and it takes the form $(\forall\xi)\,\beta[\xi]$, the wff $\beta[e_x]$ is also closed.
- By DT, $\Gamma_n \vdash \neg\beta[e_x] \Rightarrow (\delta \wedge \neg\delta)$.
- $\vdash \neg(\delta \wedge \neg\delta)$ and so by SL, $\Gamma_n \vdash \beta[e_x]$.
- 'e_x' is not included in Γ_n. Hence, we can replace 'e_x' with the variable ξ throughout the proof for $\Gamma_n \vdash \beta[e_x]$ and the result will also be a proof. Hence $\Gamma_n \vdash \beta[\xi]$.
- By Gen, $\Gamma_n \vdash (\forall\xi)\,\beta[\xi]$, which is the same as $\Gamma_n \vdash \alpha_{n+1}$.
- So Γ_n is inconsistent, which contradicts the inductive hypothesis.
- Hence Γ_{n+1} is consistent.

[†] If this assumption is not warranted, then we'll use some other denumerable sequence of constants—e.g., the '*b*'s, or '*c*'s,—or, if need be, we'll even expand the syntax of the language by adding a completely new sequence of constants, '*o*', 'o_1', 'o_2', 'o_3', …, etc.

7. It follows from (6) that Δ is *consistent*.

a) Note that the Γ-sequence is constantly *expanding*: For all j and k such that $j < k$, $\Gamma_j \subseteq \Gamma_k$. Crudely, Δ can be thought of as the upper limit of the expansion.
c) So every *finite* subset of Δ is a subset of some Γ_i for some suitably large i.
c) However, every proof from Δ has only a finite number of steps, and hence only makes use of a finite subset of Δ.
d) If there were some β such that both $\Delta \vdash \beta$ and $\Delta \vdash \neg\beta$, for some suitably large i, it would have to be that both $\Gamma_i \vdash \beta$ and $\Gamma_i \vdash \neg\beta$.
e) This is impossible because all the members of the Γ-sequence are consistent by (6).
f) Hence, Δ is consistent.
g) This establishes part (b) of the consequent of the Lemma.

8. Δ is obviously *maximal* as well.

a) All closed wffs are members of the sequence $\alpha_1, \alpha_2, \ldots$, etc.
b) For each α_i, either it or its negation is a member of Γ_i, and $\Gamma_i \subseteq \Delta$.
c) So for all closed wffs $\alpha_1, \alpha_2, \ldots$, etc., either it or its negation is included in Δ.
d) This establishes part (c) of the consequent of the Lemma.

9. Finally, Δ is also *universal*.

a) We show this by *reductio*. Suppose otherwise, i.e., suppose that there is a wff $\beta[\xi]$ that contains at most ξ free, such that for all closed terms τ, $\beta[\tau] \in \Delta$, but $(\forall\xi)\beta[\xi] \notin \Delta$.
b) $(\forall\xi)\beta[\xi]$ is closed, so because Δ is maximal, it must be that $\neg(\forall\xi)\beta[\xi] \in \Delta$.
c) Because $(\forall\xi)\beta[\xi]$ is closed, it also follows that $(\forall\xi)\beta[\xi]$ is a member of the α-sequence, i.e., $(\forall\xi)\beta[\xi]$ is α_{n+1} for some number n.
d) Obviously, however, since $(\forall\xi)\beta[\xi] \notin \Delta$, it follows that Γ_{n+1} is not obtained from Γ_n using case (i).
e) Nor was it obtained using case (ii), since α_{n+1} is of the form $(\forall\xi)\beta[\xi]$.
f) This leaves case (iii), so Γ_{n+1} is $\Gamma_n \cup \{\neg\alpha_{n+1}\} \cup \{\neg\beta[e_x]\}$.
g) Hence for some x, $\neg\beta[e_x] \in \Gamma_{n+1}$ and so $\neg\beta[e_x] \in \Delta$.
h) But by our assumption, it holds for all closed terms τ that $\beta[\tau] \in \Delta$.
i) All constants, e_x included, are closed terms, so $\beta[e_x] \in \Delta$.
j) Hence, both $\Delta \vdash \neg\beta[e_x]$ and $\Delta \vdash \beta[e_x]$.
k) However, this is impossible, because we have already shown Δ to be consistent.
l) Our supposition has been shown to be impossible, hence Δ is universal.
m) This establishes part (d) of the consequent of the Lemma.

10. By suitably defining Δ, we have shown each of parts (a)-(d) of the consequent of the Lemma on the basis of the assumption of its antecedent. Hence, the Lemma is established.

- What have we just shown? We've shown that beginning with any consistent set of sentences, we can keep adding to it *ad infinitum* to get a maximally consistent set of sentences of the language.

We pause again for a new definition:

Definition: A model or interpretation M is a ***denumerable model*** iff its domain of quantification D is denumerable (as defined on the set theory handout, p. 5).

Reminder:
An interpretation M is ***a model for*** a set of wffs Γ iff for all wffs α, if $\alpha \in \Gamma$, then $\vDash_M \alpha$.

The Maximal Consistency Lemma: If Δ is a consistent, maximal, and universal set of closed wffs, then there is at least one denumerable model for Δ.

Proof:

1. Assume that Δ is a consistent, maximal, and universal set of closed wffs. We can then describe a denumerable model M for Δ using the following procedure.

2. Essentially, we'll let all the closed terms of the language stand for *themselves*. (Another possible way of constructing a model would be to let each closed term stand for its Gödel number. However, let us proceed using the former method.)

3. Let the domain of quantification D of M be the set of closed terms of the language of first-order predicate logic. Note that there are denumerably many closed terms, so M is a denumerable model.

4. For each constant κ, let $(\kappa)^{\mathrm{M}}$ be κ itself. So, for example, (‘a’)$^{\mathrm{M}}$ is ‘a’, (‘b_{12}’)$^{\mathrm{M}}$ is ‘b_{12}’, etc.

5. For each function letter ϕ with superscript n, let $(\phi)^{\mathrm{M}}$ be that n-place operation on D which includes all ordered pairs of the form $\langle\langle\tau_1, \ldots, \tau_n\rangle, \phi(\tau_1, \ldots, \tau_n)\rangle$, i.e., the operation that has the closed term $\phi(\tau_1, \ldots, \tau_n)$ as value for $\langle\tau_1, \ldots, \tau_n\rangle$ as argument.

So, for example, the operation (‘f^1’)$^{\mathrm{M}}$, which M assigns to the monadic function letter ‘f^1’, will contain such ordered pairs as $\langle$‘a’, “$f^1(a)$”$\rangle$, $\langle$‘b_{12}’, “$f^1(b_{12})$”$\rangle$, and $\langle$“$f^1(a)$”, “$f^1(f^1(a))$”$\rangle$, and so on.

6. For each predicate letter π with superscript n, let $(\pi)^{\mathrm{M}}$ be that subset of D^n that includes the n-tuple $\langle\tau_1, \ldots, \tau_n\rangle$ iff the atomic wff $\pi(\tau_1, \ldots, \tau_n)$ is included in Δ.

So, for example, the extension of ‘F^1’ under M, viz., (‘F^1’)$^{\mathrm{M}}$, will include the term ‘a’ just in case “$F^1(a)$” $\in \Delta$, and will exclude ‘a’ just in case “$\neg F^1(a)$” $\in \Delta$, and so on.

7. We must now prove that this interpretation M is a model *for* Δ, i.e., that for all wffs α, if $\alpha \in \Delta$, then $\vDash_{\mathrm{M}} \alpha$. We will actually prove something stronger, i.e., that for all closed wffs α, $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$. ($\Delta$ only contains closed wffs, so we need not worry about open wffs.) We prove this by wff induction.

Base step: α is a closed atomic formula. Hence, α takes the form $\pi(\tau_1, \ldots, \tau_n)$ where π is a predicate letter with superscript n and $\tau_1, \ldots, \tau_n$ are closed terms.

- Because closed terms contain no variables, all sequences in M will associate each τ_i with itself.
- So by the defn. of satisfaction, all sequences in M will satisfy α iff $\langle\tau_1, \ldots, \tau_n\rangle \in (\pi)^{\mathrm{M}}$.
- So by the definition of truth in an interpretation, $\vDash_{\mathrm{M}} \alpha$ iff $\langle\tau_1, \ldots, \tau_n\rangle \in (\pi)^{\mathrm{M}}$.
- By our characterization of M under point (6) above, $\langle\tau_1, \ldots, \tau_n\rangle \in (\pi)^{\mathrm{M}}$ iff $\pi(\tau_1, \ldots, \tau_n) \in \Delta$.
- So $\pi(\tau_1, \ldots, \tau_n) \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$, i.e., $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$.

Induction step: Assume as inductive hypothesis that it holds for all closed wffs β with fewer connectives than α, that $\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \beta$. We will then show that it holds for the complex closed wff α that $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$.

This proceeds by a proof by cases on the make-up of α.

Case (a): α takes the form $\neg\beta$, where β is also closed and has one less connective than α.

- By the inductive hypothesis, $\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \beta$.
- Because Δ is consistent, if $\alpha \in \Delta$, then $\beta \notin \Delta$.
- Because Δ is maximal, if $\beta \notin \Delta$, then $\alpha \in \Delta$.
- So $\beta \notin \Delta$ iff $\alpha \in \Delta$.
- Hence $\alpha \in \Delta$ iff not-$\vDash_{\mathrm{M}} \beta$.
- Since β is closed, $\vDash_{\mathrm{M}} \neg\beta$ iff not-$\vDash_{\mathrm{M}} \beta$.
- Hence, $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \neg\beta$, i.e., $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$.

Case (b): α takes the form $\beta \Rightarrow \delta$, where β and δ are closed wffs with fewer connectives.

(i) First we prove that if $\alpha \in \Delta$ then $\vDash_{\mathrm{M}} \alpha$.

- Suppose $\alpha \in \Delta$.
- Since Δ is maximal and consistent, $\beta \in \Delta$ or $\neg\beta \in \Delta$, but not both, and likewise with δ.
- However, because $\beta \Rightarrow \delta \in \Delta$ and Δ is consistent, it cannot be that both $\beta \in \Delta$ and $\neg\delta \in \Delta$, so either $\neg\beta \in \Delta$ or $\delta \in \Delta$.
- By the inductive hypothesis, $\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \beta$, and $\delta \in \Delta$ iff $\vDash_{\mathrm{M}} \delta$.
- By the same reasoning given for case (a), $\neg\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \neg\beta$.
- So either $\vDash_{\mathrm{M}} \neg\beta$ or $\vDash_{\mathrm{M}} \delta$.
- By the definition of satisfaction for conditionals, it follows that $\vDash_{\mathrm{M}} \beta \Rightarrow \varepsilon$, i.e., $\vDash_{\mathrm{M}} \alpha$.

(ii) Now we prove that if $\vDash_{\mathrm{M}} \alpha$ then $\alpha \in \Delta$.

- Suppose $\vDash_{\mathrm{M}} \alpha$, i.e., $\vDash_{\mathrm{M}} \beta \Rightarrow \delta$.
- Because β and δ are closed, by the definition of satisfaction for conditionals, we have either $\vDash_{\mathrm{M}} \neg\beta$ or $\vDash_{\mathrm{M}} \delta$.
- By the inductive hypothesis, $\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \beta$ and $\delta \in \Delta$ iff $\vDash_{\mathrm{M}} \delta$.
- Again, by the reasoning given for case (a), $\neg\beta \in \Delta$ iff $\vDash_{\mathrm{M}} \neg\beta$.
- So either $\neg\beta \in \Delta$ or $\delta \in \Delta$.
- Because Δ is maximal, either $\beta \Rightarrow \delta \in \Delta$ or $\neg(\beta \Rightarrow \delta) \in \Delta$.
- If $\neg(\beta \Rightarrow \delta) \in \Delta$, then Δ would be inconsistent, because $\vdash \neg(\beta \Rightarrow \delta) \Rightarrow \beta$ and $\vdash \neg(\beta \Rightarrow \delta) \Rightarrow \neg\delta$.
- So $\beta \Rightarrow \delta \in \Delta$, i.e., $\alpha \in \Delta$.

(iii) Putting (i) and (ii) together, we get that $\alpha \in \Delta$ iff $\vDash_{\mathrm{M}} \alpha$.

Case (c): α takes the form $(\forall\xi)\beta[\xi]$, where $\beta[\xi]$ contains fewer connectives, and $\beta[\xi]$ contains at most ξ free.

(i) First we prove that if $\alpha \in \Delta$ then $\vDash_M \alpha$.

- Suppose $\alpha \in \Delta$, i.e., $(\forall\xi)\beta[\xi] \in \Delta$.
- Because $\beta[\xi]$ contains at most ξ free, for all closed terms τ, $\beta[\tau]$ is a closed wff.
- Because Δ is maximal, for all closed terms τ, either $\beta[\tau] \in \Delta$ or $\neg\beta[\tau] \in \Delta$.
- However, since Δ is consistent, it must be that for all closed terms τ, $\beta[\tau] \in \Delta$.
- By the inductive hypothesis, for all closed terms τ, $\vDash_M \beta[\tau]$.
- Because the domain of quantification for M is *D*, and *D* consists of the set of closed terms, and every closed term is interpreted as standing for itself, a sequence of M will satisfy $\beta[\xi]$ iff it satisfies $\beta[\tau]$ for that closed term τ that gets assigned to ξ in that sequence.
- Because all sequences of M satisfy $\beta[\tau]$ for all closed terms τ, all sequences of M will satisfy $\beta[\xi]$, and hence all sequences of M will satisfy $(\forall\xi)\beta[\xi]$.
- Hence, $\vDash_M (\forall\xi)\beta[\xi]$, i.e., $\vDash_M \alpha$.

(ii) We now prove that if $\vDash_M \alpha$ then $\alpha \in \Delta$.

- Suppose $\vDash_M \alpha$, i.e., all sequences of M satisfy $(\forall\xi)\beta[\xi]$.
- Hence, all sequences of M satisfy $\beta[\xi]$, regardless of what entity in the domain gets assigned to ξ.
- Because the domain of quantification for M is *D*, and *D* consists of the set of closed terms, and every closed term is interpreted as standing for itself, a sequence of M will satisfy $\beta[\xi]$ iff it satisfies $\beta[\tau]$ for that closed term τ that gets assigned to ξ in that sequence.
- So, for all closed terms τ, $\vDash_M \beta[\tau]$.
- By the inductive hypothesis, it follows that, for all closed terms τ, $\beta[\tau] \in \Delta$.
- Because Δ is universal, it follows that $(\forall\xi)\beta[\xi] \in \Delta$, i.e., $\alpha \in \Delta$.

(iii) Putting (i) and (ii) together, we get that $\alpha \in \Delta$ iff $\vDash_M \alpha$.

9. By induction, regardless of α's length, $\alpha \in \Delta$ iff $\vDash_M \alpha$. So M is a model for Δ. This establishes the Lemma.

- If we follow Mendelson and think of a model as a sort of possible world, a maximally consistent set of sentences can be thought of as a maximally descriptive yet consistent description of a possible world. This lemma says that for every maximally descriptive consistent description of a possible world, one exists for which that description is true.

The Modeling Lemma: A set of closed wffs Γ is consistent iff it has a denumerable model (i.e., there is at least one denumerable model for Γ).

Proof:

This biconditional breaks down into:

(MLa) If a set of closed wffs Γ has a denumerable model, then Γ is consistent.

(MLb) If a set of closed wffs Γ is consistent, then Γ has a denumerable model.

Instead of proving (MLa), we shall prove the following *stronger* thesis:

> (MLa)* If a set of closed wffs Γ has any model, then Γ is consistent.

Proof of (MLa)*:

1. Assume the opposite for *reductio ad absurdum*. I.e., assume that Γ is a set of closed wffs, and there is at least one model M for Γ, but that Γ is inconsistent.
2. Hence, there is some α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.
3. This means that α and $\neg\alpha$ are each derivable from the members of Γ along with the axioms of PF by zero or more applications of MP and Gen.
4. All the axioms of PF are logically valid, and hence true in M.
5. Similarly, all the members of Γ are true in M by hypothesis.
6. However, both MP and Gen preserve truth in an interpretation, so it must be that both $\vDash_M \alpha$ and $\vDash_M \neg\alpha$.
7. By the definition of truth in an interpretation, every sequence in M satisfies both α and $\neg\alpha$.
8. However, a sequence satisfies $\neg\alpha$ iff it does not satisfy α, so any arbitrary sequence of M will both satisfy and not satisfy α, which is absurd.
9. Hence (MLa)* must be true. Regardless of the size of the domain, any set of closed wffs that can be modeled is consistent. This includes those with denumerable models, so (MLa)* entails (MLa).

Proof of (MLb):

1. Assume that Γ is a consistent set of closed wffs.
2. By the Lindenbaum Extension Lemma, there is a set of closed wffs Δ such that: (a) $\Gamma \subseteq \Delta$, (b) Δ is consistent, (c) Δ is maximal, and (d) Δ is universal.
3. By the Maximal Consistency Lemma, there is an interpretation M that is a denumerable model for Δ.
4. So for all closed wffs α, if $\alpha \in \Delta$, then $\vDash_M \alpha$.
5. Because $\Gamma \subseteq \Delta$, for all closed wffs α, if $\alpha \in \Gamma$ then $\alpha \in \Delta$.
6. So for all closed wffs α, if $\alpha \in \Gamma$ then $\vDash_M \alpha$.
7. Therefore, M is also a denumerable model for Γ.

The following is not needed for completeness, but is an interesting and surprising result of (MLa)* and (MLb):

The Skolem-Löwenheim Theorem: If a set Γ of closed wffs of first-order predicate logic has any sort of model, then it has a denumerable model

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By the stronger (MLa)*, if Γ has any sort of model, then it is consistent. By (MLb), if it is consistent, it has a *denumerable* model.

Finally we turn to completeness:

Completeness: For all wffs α, if $\vDash \alpha$ then $\vdash \alpha$.

Proof:

1. Suppose $\vDash \alpha$, but suppose for *reductio ad absurdum* that it is not the case that $\vdash \alpha$.
2. Let β be the universal closure of α, i.e., if the free variables of α are $\xi_1, \ldots, \xi_n$, then β is $(\forall\xi_1)\ldots(\forall\xi_n)\alpha$.
3. Universal closure preserves truth in an interpretation, so $\vDash \beta$.
4. β has no free variables left, so β is closed.
5. The singleton set containing $\neg\beta$ alone, $\{\neg\beta\}$, must be consistent. Here's a proof of this by *reductio*:
    a. Suppose there were some δ such that $\{\neg\beta\} \vdash \delta$ and $\{\neg\beta\} \vdash \neg\delta$.
    b. By SL, $\{\neg\beta\} \vdash \delta \wedge \neg\delta$.
    c. Since β is closed, so is $\neg\beta$, and so by DT, we have $\vdash \neg\beta \Rightarrow (\delta \wedge \neg\delta)$.
    d. But $\vdash \neg(\delta \wedge \neg\delta)$, so by SL, $\vdash \beta$.
    e. But α is derivable from β by universal instantiation, so it would follow that $\vdash \alpha$, which contradicts our earlier assumption.
    f. Hence, $\{\neg\beta\}$ is consistent.
6. By the Modeling Lemma, $\{\neg\beta\}$ has a denumerable model. Hence there is an interpretation M such that $\vDash_M \neg\beta$.
7. But we also have $\vDash \beta$, and hence $\vDash_M \beta$.
8. By the definition of truth in an interpretation, every sequence in M satisfies both β and $\neg\beta$.
9. However, a sequence satisfies $\neg\beta$ iff it does not satisfy β, so any arbitrary sequence of M will both satisfy and not satisfy β, which is absurd.
10. We've shown our supposition to be impossible, thereby establishing completeness indirectly.

Unfortunately, this proof does not, as in the Propositional Calculus (System L), provide a "recipe" for constructing a proof of any given logical truth in PF. We have simply proven that any given logical truth must be derivable, because if it were not, there would exist a countermodel to its logical validity.

The completeness of the first-order predicate calculus was first proven by Kurt Gödel in 1930, and so this is sometimes called "Gödel's Completeness Theorem," although his way of proving it was actually very different than ours. (It was first proven *our way* by Leon Henkin in 1949.) However, Gödel is much more famous for his *incompleteness* theorems than his completeness theorem.