“Proposition 1.2”
For any wffs α and β, if $\vDash \alpha$ and $\vDash (\alpha \Rightarrow \beta)$ then $\vDash \beta$.

First, to be clear about what we’re doing, we’re not proving that *modus ponens* is a valid reasoning form. That would be to prove that

$$\{(\alpha \Rightarrow \beta), \alpha\} \vDash \beta$$

The above is true, and easily proven, but it’s not what we’re after. Instead, we’re proving something a bit stronger, namely that *modus ponens preserves tautologyhood*, i.e., that if both α and $\alpha \Rightarrow \beta$ are *tautologies*, then β is a tautology as well.

What we’re proving is a conditional. We assume the antecedent and attempt to prove the consequent.

1. Assume that $\vDash \alpha$ and $\vDash (\alpha \Rightarrow \beta)$.
2. What we’ve assumed in line 1 means that both α and $(\alpha \Rightarrow \beta)$ are tautologies, i.e., that every possible truth-value assignment to the statement letters making them up makes them true.
3. Suppose for *reductio ad absurdum* that there were some truth-value assignment (row of a truth table) making β false.
4. Notice that because every truth-value assignment makes $(\alpha \Rightarrow \beta)$ true, if it makes β false it must make α false as well.
5. From lines 3 and 4 we get the result that there is a truth-value assignment making α false.
6. However, it follows from line 2 that no truth-value assignment makes α false.
7. Lines 5 and 6 are a contradiction, and so our assumption at line 3 is false, and so $\vDash \beta$.
8. Therefore, by conditional proof, if $\vDash \alpha$ and $\vDash (\alpha \Rightarrow \beta)$ then $\vDash \beta$. QED.

In your book, you’ll also find proofs of the following results:

“Proposition 1.3”: if $\vDash \alpha$, and β is the result of (uniformly) replacing certain statement letters in α by complex wffs, then $\vDash \beta$.

“Proposition 1.4”: If α is a wff containing wff δ in one or more places, and β is just like α except containing wff γ in those places where α contains δ, then if $\delta \dashv\vDash \gamma$ then $\alpha \dashv\vDash \beta$.

VII. Reducing the Number of Connectives

When working *within* a given language, usually the more complex it is, the easier it is to say what you want, because you have more vocabulary in which to say it. However, when you’re trying to prove something *about* the language, it’s usually easier if it is simpler, because the more complex the language is, the more there is *to say* about it. When doing logical meta-theory, it’s usually to our advantage to whittle down our object language (and the logical calculi we develop in it) to as small as possible. To that end, we ask, do we really need all five connectives ($\neg$, $\Rightarrow$, $\Leftrightarrow$, $\wedge$ and $\vee$)?

After all, our object language is not inadequate in any way by not including a sign for the *exclusive* sense of disjunction, since we can represent it using other signs, e.g., as $(\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta)$ or $\neg(\alpha \Leftrightarrow \beta)$, etc. And no, we don’t need all five of the ones we have. First we’ll show that we could get by with just three, and later two, and finally one.

Result: Every possible truth function can be represented by means of the connectives ‘$\wedge$’, ‘$\vee$’ and ‘$\neg$’ alone.

We’ll prove this somewhat informally:
1. Assume that α is some wff built using any set of truth-functional connectives, including, if you like, connectives other than our five. For example, α might make use of some three or four-place truth-functional connectives, or connectives such as the exclusive or, or any others you might imagine for bivalent logic.
2. What we’re going to show is that there is a wff β formed only with the connectives ‘$\wedge$’, ‘$\vee$’ and ‘$\neg$’ that is logically equivalent with α.
3. In order for it to be logically equivalent to α, the wff β that we construct must have the same final truth value for every possible truth-value assignment to the statement letters making up α, or in other words, it must have the same final column in a truth table.
4. Let ρ_1, ρ_2, ..., ρ_n be the distinct statement letters making up α. For some possible truth-value assignments to these letters, α may be true, and for others α may be false. The only hard case would be the one in which α is contingent. If α is not contingent, it must either be a tautology, or a self-contradiction. Since clearly tautologies and self-contradictions can

be constructed with the signs '∧', '∨' and '¬', and all tautologies are logically equivalent to one another, and all self-contradictions are equivalent to one another, in those cases, our job is easy. Let us suppose instead that α is contingent.

5. Let us construct a wff β in the following way.

(a) Consider in turn each possible truth-value assignment to the letters $\rho_1, \rho_2, \ldots, \rho_n$. For each truth-value assignment, construct a conjunction made up of those letters the truth-value assignment makes true, along with the negations of letters the truth-value assignment makes false.

Example: Suppose the letters involved are 'A', 'B' and 'C'. This means that there are eight possible truth-value assignments, corresponding to the eight rows of a truth table. We construct an appropriate conjunction for each.

A	*B*	*C*	
T	T	T	$A \land B \land C$
T	T	F	$A \land B \land \neg C$
T	F	T	$A \land \neg B \land C$
T	F	F	$A \land \neg B \land \neg C$
F	T	T	$\neg A \land B \land C$
F	T	F	$\neg A \land B \land \neg C$
F	F	T	$\neg A \land \neg B \land C$
F	F	F	$\neg A \land \neg B \land \neg C$

(b) From the resulting conjunctions, form a complex disjunction formed from those conjunctions formed in step (a) for which the corresponding truth-value assignment makes α true.

Example: Suppose for the example above that the final column of the truth table for α is as follows (just at random):

A	*B*	*C*	α
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

This means that we form a disjunction using as the disjuncts those conjunctions formed in step (a) for those rows that make α true. The others are left out. In this case, our result is:

$(A \land B \land C) \lor (A \land B \land \neg C) \lor (\neg A \land B \land C)$

The three conjunctions in the disjunction conform to the three truth-value assignments that make α true.

6. The wff β constructed in step 5 is logically equivalent to α. Consider that for those truth-value assignments making α true, one of the conjunctions making up the disjunction β is true, and hence the whole disjunction is true as well. For those truth-value assignments making α false, none of the conjunctions making up β is true, because each conjunction will contain at least one conjunct that is false on that truth-value assignment.

Example: Let us construct a truth table for the formula we constructed during our last step:

(A	∧	B	∧	C)	∨	(A	∧	B	∧	¬	C)	∨	(¬	A	∧	B	∧	C)
T	T	T	T	T	T	T	T	T	F	F	T	**T**	F	T	F	T	F	T
T	T	T	F	F	T	T	T	T	T	T	F	**T**	F	T	F	T	F	F
T	F	F	F	T	F	T	F	F	F	F	T	**F**	F	T	F	F	F	T
T	F	F	F	F	F	T	F	F	F	T	F	**F**	F	T	F	F	F	F
F	F	T	F	T	F	F	F	T	F	F	T	**T**	T	F	T	T	T	T
F	F	T	F	F	F	F	F	T	F	T	F	**F**	T	F	T	T	F	F
F	F	F	F	T	F	F	F	F	F	F	T	**F**	T	F	F	F	F	T
F	F	F	F	F	F	F	F	F	F	T	F	**F**	T	F	F	F	F	F

By examining the final column for this truth table, we see that it has the same final column as that given for α.

This establishes our result. The example was arbitrary; the same process would work regardless of the number of statement letters or final column for the statement involved.

Reducing Further

The above result means that any set of connectives in which we can always find equivalent forms for (α ∨ β), (α ∧ β) and ¬α is an adequate set of connectives. This means we can reduce still further. We don't need all three. We can get by with two in any of three ways.

(1) We could get by with just $\neg$ and $\vee$. Notice that $\neg(\neg\alpha \vee \neg\beta)$ is equivalent to $(\alpha \wedge \beta)$ and could be used instead.

(2) We could get by with just $\neg$ and $\wedge$. Notice that $\neg(\neg\alpha \wedge \neg\beta)$ is equivalent with $(\alpha \vee \beta)$ and could be used instead.

(3) We could get by with just $\neg$ and $\Rightarrow$. Notice:

$(\neg\alpha \Rightarrow \beta) \dashv\vDash (\alpha \vee \beta)$

$\neg(\alpha \Rightarrow \neg\beta) \dashv\vDash (\alpha \wedge \beta)$

So we could take neither $\vee$ nor $\wedge$ as primitive and instead simply use the forms above.

Reducing Still Further

Actually, if we started from a different basis, we could get by with just one connective.

(4) The most common way to do this is with the *Sheffer stroke*, written '|'. It has the following truth table:

α	β	(α | β)
T	T	F
T	F	T
F	T	T
F	F	T

A statement of the form $(\alpha \mid \beta)$ is true in every case except that in which α and β are both true. So "$\alpha \mid \beta$" could be read "not both α and β", and indeed is equivalent to $\neg(\alpha \wedge \beta)$. However, since our aim is to reduce all operators to '|', it is best not to think of the meanings of '$\neg$' or '$\wedge$' as playing a role.

The Sheffer stroke on its own is adequate, since:

$(\alpha \mid \alpha) \dashv\vDash \neg\alpha$

$((\alpha \mid \alpha) \mid (\beta \mid \beta)) \dashv\vDash (\alpha \vee \beta)$

$((\alpha \mid \beta) \mid (\alpha \mid \beta)) \dashv\vDash (\alpha \wedge \beta)$

and just for kicks, we can add:

$(\alpha \mid (\beta \mid \beta)) \dashv\vDash (\alpha \Rightarrow \beta)$

$(((\alpha \mid \alpha) \mid (\beta \mid \beta)) \mid (\alpha \mid \beta)) \dashv\vDash (\alpha \Leftrightarrow \beta)$

(5) Another way is with the *Sheffer/Peirce dagger*, written '↓', which has the truth table:

α	β	(α ↓ β)
T	T	F
T	F	F
F	T	F
F	F	T

"$\alpha \downarrow \beta$" might be read as "neither α nor β".

We can get all the others, since:

$(\alpha \downarrow \alpha) \dashv\vDash \neg\alpha$

$((\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta)) \dashv\vDash (\alpha \wedge \beta)$

$((\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta)) \dashv\vDash (\alpha \vee \beta)$

But that's it. '|' and '↓' are the only binary connectives from which all truth functions can be derived. In fact, we can prove this.

Result: No binary operator besides '|' and '↓' is by itself sufficient to capture all truth functions.

Proof:

1. Suppose there were some other binary connective # that was adequate by itself.
2. We know immediately that (α # β) must be false when α and β are both true. If not, then it would impossible to form something equivalent to a contradiction, since the "top row" of the truth table (the truth-value assignment making all statement letters true) would always make a wff true. For similar reasons, (α # β) must be true when α and β are both false, or else it would be impossible to form something equivalent to a tautology.
3. Line 2 gives us this much of the table for #:

α	β	α # β
T	T	F
T	F	?
F	T	?
F	F	T

The question is how to fill in the remaining ?'s.

4. If we fill both in with T's get the Sheffer stroke. If we fill both in F's, we get the *Sheffer/Peirce dagger*. That rules out two of the four remaining possibilities.
5. If we fill them in T and F respectively, the result is equivalent with $\neg\beta$, and if we fill them in with F and T, the result is equivalent with $\neg\alpha$.
6. Negation is clearly insufficient for defining all other truth functions. So the remaining options in step 5 are inadequate. There are no possibilities left. Our # is impossible. QED.

There are, however, triadic connectives and 4+ place connectives that work.