IV. An Axiom System

As our system of deduction for predicate logic, we introduce the following. (Below, 'α', 'β', and 'δ' are used as schematic letters representing wffs, 'ξ' as a schematic letter for individual variables, and 'τ' for terms. 'Γ' is used as a metalinguistic variable ranging over *sets* of object-language wffs.)

The First-Order Predicate Calculus (System PF):[†]

Definition: An ***axiom*** of PF is any wff of one of the following five forms:

(A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$
(A2) $(\alpha \Rightarrow (\beta \Rightarrow \delta)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \delta))$
(A3) $(\neg\alpha \Rightarrow \neg\beta) \Rightarrow ((\neg\alpha \Rightarrow \beta) \Rightarrow \alpha)$
(A4) $(\forall\xi)\alpha[\xi] \Rightarrow \alpha[\tau]$, for all instances such that τ is free for ξ in $\alpha[\xi]$
(A5) $(\forall\xi)(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow (\forall\xi)\beta)$, for all instances such that α contains no free occurrences of ξ.

The ***inference rules*** of PF are:

Modus ponens (MP): From $(\alpha \Rightarrow \beta)$ and α, infer β.
Generalization (Gen): From α infer $(\forall\xi)\alpha$.

"$\Gamma \vdash \alpha$" and simply "$\vdash \alpha$" are defined as you might expect. In this unit, unless otherwise specified, '$\vdash$' means '$\vdash_{PF}$'.

Definition: A ***first-order theory*** K is an axiomatic system in the language of predicate logic that can be obtained from the above by adding zero or more ***proper** or **non-logical axioms***. Proper axioms are added to represent the basic principles of a certain area of thought. E.g., we might form a first-order theory for the study of the solar system by using the constants 'a_1', …, 'a_9' as the nine planets, 'b_1' for the sun, etc., using 'O^2', for the orbiting relation, etc., and adding as axioms certain laws of physics stated in the language of predicate logic, and so on.

Note that every theorem schema of L corresponds to a theorem schema of PF (or any other first-order theory). Since L is complete, if α is a truth-table tautology, then $\vdash_{PF} \alpha$. You may cite this in your proofs by writing "Taut" as justification. Similarly, every derived rule of L corresponds to a derived rule of PF. You may make use of this in your derivations by using the abbreviations on p. 20, or simply writing "SL" [System L] as justification. Alternatively, you may utilize the notation used within your favorite natural deduction system for propositional logic, or abbreviate the names given with the derived rules listed on pp. 77-78 of your textbook.

<u>All first-order theories, including the bare-bones PF, have the following additional derived rules</u>:

Universal instantiation (UI, $\forall$O or *rule* A4):

$(\forall\xi)\alpha[\xi] \vdash \alpha[\tau]$, where τ is free for ξ in $\alpha[\xi]$.

Follows directly from (A4) by MP.

Existential generalization (EG or $\exists$I or E4):

$\alpha[\tau, \tau] \vdash (\exists\xi)\alpha[\tau, \xi]$, where τ is free for ξ in $\alpha[\tau, \xi]$.

Proof schema:

1. $\alpha[\tau, \tau] \vdash \alpha[\tau, \tau]$	(Premise)
2. $\alpha[\tau, \tau] \vdash \neg\neg\alpha[\tau, \tau]$	1 SL (DN)
3. $\vdash (\forall\xi)\neg\alpha[\tau, \xi] \Rightarrow \neg\alpha[\tau, \tau]$	A4
4. $\alpha[\tau, \tau] \vdash \neg(\forall\xi)\neg\alpha[\tau, \xi]$	2, 3 SL (MT)
5. $\alpha[\tau, \tau] \vdash (\exists\xi)\alpha[\tau, \xi]$	4 definition of $\exists$

Replacement of free variables (Sub or Repl):

$\alpha[\xi] \vdash \alpha[\tau]$, where τ is free for ξ in $\alpha[\xi]$.

Proof schema:

1. $\alpha[\xi] \vdash \alpha[\xi]$	(Premise)
2. $\alpha[\xi] \vdash (\forall\xi)\alpha[\xi]$	1 Gen
3. $\alpha[\xi] \vdash \alpha[\tau]$	2 UI

[†] PF stands for "*Full* Predicate calculus", i.e., the calculus within a syntax including all possible constants, predicate letters and function letters. The "*Pure* Predicate calculus", PP, is the same, but excluding all constants or function letters from the syntax These abbreviations are given on p. 221 of your book, in Chapter 3. There are predicate calculi that are neither pure nor full.

V. The Deduction Theorem in PF

The deduction theorem does not hold generally in the first-order predicate calculus PF, nor would we want it to. After all, in the semantics of predicate logic, it is not the case that $\vDash Fx \Rightarrow (\forall x)Fx$, and similarly in the system of deduction, while we have $Fx \vdash (\forall x)Fx$ by Gen we should not have $\vdash Fx \Rightarrow (\forall x)Fx$. We therefore state and prove the deduction theorem in the following restricted form:

(DT) If $\Gamma \cup \{\delta\} \vdash \alpha$ and in the proof $\beta_1, \ldots, \beta_n$ of α from $\Gamma \cup \{\delta\}$, no step is obtained by an application of Gen that *both* (i) is applied to a previous step that depends upon having δ in the premise set, and (ii) uses a variable occurring free in δ, then $\Gamma \vdash \delta \Rightarrow \alpha$.

Proof:

1. Assume the complex antecedent of DT. We will show, using *proof induction*, that for every step β_i in the proof $\beta_1, \ldots, \beta_n$ of α from $\Gamma \cup \{\delta\}$, that it holds that $\Gamma \vdash \delta \Rightarrow \beta_i$. We are entitled to assume that we have already gotten $\Gamma \vdash \delta \Rightarrow \beta_j$ for all steps β_j prior to β_i.

2. Because β_i is a step in the proof of α from $\Gamma \cup \{\delta\}$, the cases we have to consider are that: (a) β_i is a member of Γ, (b) β_i is δ, (c) β_i is an axiom, (d) β_i follows from previous steps in the proof by MP, and (e) β_i follows from a previous step by an application of Gen obeying the restriction mentioned above. We consider each case.

Case (a). β_i is a member of Γ. Hence $\Gamma \vdash \beta_i$, and by SL, $\Gamma \vdash \delta \Rightarrow \beta_i$.

Case (b). β_i is δ. Then $\delta \Rightarrow \beta_i$ is simply $\delta \Rightarrow \delta$, a simple tautology, whence $\Gamma \vdash \delta \Rightarrow \beta_i$.

Case (c). β_i is an axiom. Hence $\vdash \beta_i$ and by SL, obtain $\vdash \delta \Rightarrow \beta_i$. *A fortiori*, $\Gamma \vdash \delta \Rightarrow \beta_i$.

Case (d). β_i follows from previous members of the series by MP. Therefore there are previous members of the series β_j and β_k such that β_j takes the form $\beta_k \Rightarrow \beta_i$. By the inductive hypothesis, we already have both $\Gamma \vdash \delta \Rightarrow \beta_k$ and $\Gamma \vdash \delta \Rightarrow (\beta_k \Rightarrow \beta_i)$. By SL, $\Gamma \vdash \delta \Rightarrow \beta_i$.

Case (e). β_i follows from a previous member of the series by an application of Gen obeying the restriction mentioned above. Therefore, there is a previous step β_j such that β_i takes the form $(\forall \xi)\, \beta_j$ for some variable ξ. Because of the restriction, either obtaining β_j did not depend on having δ in the premise set, or δ does not contain ξ free. In the first sub-case, $\Gamma \vdash \beta_j$ and hence by Gen, we have $\Gamma \vdash (\forall \xi)\, \beta_j$, i.e., $\Gamma \vdash \beta_i$. By SL, then, $\Gamma \vdash \delta \Rightarrow \beta_i$, as usual. In the second sub-case, we first note that we have $\Gamma \vdash \delta \Rightarrow \beta_j$ by the inductive hypothesis. By Gen, we obtain $\Gamma \vdash (\forall \xi)\, (\delta \Rightarrow \beta_j)$. Because δ does not contain ξ free, as an instance of (A5) we have $\vdash (\forall \xi)\, (\delta \Rightarrow \beta_j) \Rightarrow (\delta \Rightarrow (\forall \xi)\, \beta_j)$. By MP, $\Gamma \vdash \delta \Rightarrow (\forall \xi)\, \beta_j$, i.e., $\Gamma \vdash \delta \Rightarrow \beta_i$.

Obviously, in the proof establishing that $Fx \vdash (\forall x)Fx$, Gen is applied to a step that both depends on Fx and makes use of a variable occurring free in Fx. (Note that invoking the “Sub” or “Repl” derived rule requires the same.) So we cannot conclude $\vdash Fx \Rightarrow (\forall x)Fx$. However, such is not the case with the proof:

1.	$(\forall x)Fx \vdash (\forall x)Fx$	Premise
2.	$\vdash (\forall x)Fx \Rightarrow Fy$	(A4)
3.	$(\forall x)Fx \vdash Fy$	1, 2 MP
4.	$(\forall x)Fx \vdash (\forall y)Fy$	3 Gen

This we transform as follows:

1.	$\vdash (\forall x)Fx \Rightarrow (\forall x)Fx$	(Taut)
2.	$\vdash (\forall x)Fx \Rightarrow Fy$	A4
3.	$\vdash (\forall x)Fx \Rightarrow ((\forall x)Fx \Rightarrow Fy)$	2 SL
4.	$\vdash (\forall x)Fx \Rightarrow Fy$	1, 3 SL
5.	$\vdash (\forall y)((\forall x)Fx \Rightarrow Fy)$	4 Gen
6.	$\vdash (\forall y)((\forall x)Fx \Rightarrow Fy) \Rightarrow ((\forall x)Fx \Rightarrow (\forall y)Fy)$	A5
7.	$\vdash (\forall x)Fx \Rightarrow (\forall y)Fy$	5, 6 MP

Again, this is not the most eloquent proof. (Getting line 4 by SL is silly, since it’s an axiom, and indeed, the same one introduced at line 2.) However, that’s what case (d) called for and following the rote procedure always works.

From here on out, you can use this to shorten your proofs, but BEAR IN MIND THE RESTRICTIONS. *You need to make sure that you don’t apply it when you’ve used Gen on a variable appearing free in an assumption!*