III. Cocchiarella's Higher-Order Variants

A. Background

Nino Cocchiarella has attempted what he calls a "reconstruction" of logicism based on a higher-order system employing a notion of stratification at least partly inspired by Quine.

The other part of motivation comes from the work of Frege (and very early Russell). According to the way Frege thinks language works, some expressions refer to objects, and others to concepts. Names like "Plato", "Berlin" and descriptions, e.g., "the largest star in the Big Dipper", refer to objects. The remainder of a sentence, when a referring expression such as a name is removed, refers to a *concept*.

____ is a horse.
____ orbits the Sun.

Frege thought that both such expressions, and *what they refer to* (concepts) are in some sense "incomplete" or "unsaturated".

The above are examples of sentences from which a name has been removed. Such expressions refer to what Frege calls *first-level concepts*, or concepts applicable to objects. If one removes the name of a first-level concept from a sentence, one is left with the name of a *second-level concept*.

E.g., if we remove "is a horse" from these sentences:

> Something is such that it is a horse.
> If Socrates is a horse, then Plato is a horse.

We obtain:

> Something is such that … (it) ….
> If …(Socrates)… then …(Plato)…

Depending on which name of a first-level function completes the gap, one is left with something true or something false. Second-level concepts are applicable (or not) to first-level concepts. This forms the beginning of a *hierarchy* of levels of concepts akin to what is found in higher-order simple type-theory.

Frege regarded it as impossible for a concept to be predicated of itself. The reason of course is that different kinds of concepts exhibit different kinds of incompleteness. A concept never has the right sort of incompleteness in order to "complete" (or as Frege says, "saturate" itself.)

"____ is a horse" must be completed by a name, not something such as "____ is a horse" to yield a complete, grammatical sentence. "If …(Plato)…, then …(Socrates)…" can be completed by "___ is a horse" to form something grammatical, but not by itself, etc.

This hierarchy of levels of concepts was reflected in the grammar of Frege's logical language. Different styles of variables were used for different levels, and it would be nonsense to instantiate a variable of one type to an expression of another, or place a variable of one type in its own argument spot, e.g., "$F(F)$" or "$\neg F(F)$". (Recall that for Frege, the expression for the first-level concept is not simply "F" but "$F(\ )$", revealing the incompleteness or unsaturatedness of the concept.)

The use of different styles of variables, and different "shape" expressions for different logical types in some ways sets Frege's higher-order logic apart from modern renditions that make use of lambda abstracts. If the second level concept written in everyday English as "Something is such that … (it)…." is written as "$(\exists x)$ …x…", it is quite clear that it does not "fit" in its own blank space. It is not *physically possible* to violate the type-restriction.

However, if it is written instead "$[\lambda F\ (\exists x)Fx]$", it is at least physically possible to write "$[\lambda F\ (\exists x)Fx]([\lambda F\ (\exists x)Fx])$", and one would need special *grammatical rules* (involving, e.g., type indices) to exclude such a formula.

Frege went so far in analyzing ordinary language as to suggest that an expression such as "the concept horse", because it is not an incomplete or gappy expression, and because it can fill the blank spot in an expression such as "____ is a horse", or "___ orbits the Sun" *must not*, despite appearance, refer to a concept, but to an object.

Thus Frege thought that for every concept, there was an object that could "go proxy for it", i.e., would be referred to by a nominalized predicate derived from a predicate expression for the concept. Thus, "the concept horse", "Humanity", "Kindness", etc., refer to what Cocchiarella calls "concept-correlates". There is also evidence, as Cocchiarella points out, that Frege identified these "concept-correlates" with the extension or *value-range* of a concept. Frege further claimed that the extension or value-range of a concept "had its being" in the concept: that they were in some sense, the same entity in a metaphysical sense.

For a given concept expression $F(\ )$, one refers to its value-range as "$\grave{\alpha}(F(\alpha))$". In my formulation of Frege's GG, I changed this notation to $\{x|F(x)\}$, in keeping with the usual reading of Frege's extensions of concepts as *classes*. Cocchiarella, however, suggests that Frege's logic of value-ranges can instead be thought of as a logic of *nominalized predicates*, i.e., one in which expressions for concepts can occur in both predicate and subject positions. In effect, he is suggesting that $F(\grave{\alpha}(F(\alpha)))$ is really just a variation on $F(F)$.

The effect then of including value-ranges is more or less to *undo* the effects of having distinct levels. Every first-level concept corresponds to an object: its concept-correlate (or value-range), etc. Every second-level concept can be represented by a first-level concept which applies to an object just in case that object is the correlate of a concept to which the second-level concept applies. Cocchiarella calls this Frege's "double-correlation thesis":

$(\forall M)(\exists G)(\forall F)((Mx)Fx \leftrightarrow G(\grave{\alpha}(F(\alpha))))$

By repeating this reasoning, in effect, all concepts can be "reduced in level" all the way down.

Of course, the introduction of concept-correlates as objects corresponding to *any arbitrary concept* allows one in effect to apply concepts to themselves, and thus obtain Russell's paradox in the form gotten by considering whether or not the concept whose value range is:

$\grave{\varepsilon}((\exists F)(\varepsilon = \grave{\alpha}(F(\alpha)) \wedge \neg F(\varepsilon))$

applies to that very value-range.

Cocchiarella suggests a *reconstruction* of Frege's logicism in which *some concepts* are thought to have objects corresponding to them, others not. To determine which ones do, and which ones don't, Cocchiarella modifies Quine's notion of *stratification*. Here it is used not to restrict which sets are postulated to exist, but which concepts can be "converted" into objects.

Rather than employing Frege's notation "$\grave{\varepsilon}(\ldots\varepsilon\ldots)$", or the set theoretic notation "$\{x|\ldots x\ldots\}$", Cocchiarella employs what is now the usual notation for forming complex predicates, viz. "$[\lambda x \ldots x\ldots]$". However, such an expression, *if homogenously stratified* (defined below) may occur *either* in a predicate *or* in a subject position, so that we may allow some instances of:

$[\lambda x\ \alpha(x)](\ [\lambda x\ \alpha(x)])$

This is supposed to be analogous to Frege's allowance of such constructions as "the concept horse is a horse", in which, a concept is predicated of its own concept-correlate object.

Cocchiarella's systems are formulated in the context of a second-order logic (higher orders are not necessary given the possibility of *reducing level*). Allowing suitably stratified λ-abstracts as valid substituends of both individual and predicate variables in part undoes the usual type-distinctions employed within a second, but this is no more dangerous (suggests Cocchiarella) than the effect of removing the type-restrictions of ST in favor of the stratification restrictions employed in system such as NF.

B. The System of Homogenous Simple Types (λ-HST*): the Syntax

An *individual variable* is any lowercase letter from '*u*' to '*z*' with or without a numerical subscript.

A *predicate variable* is any uppercase letter from '*A*' to '*T*', with a numerical superscript $\geqslant 1$ (indicating how many terms it is applied to), and with or without a numerical subscript.

Well-formed expressions come in two varieties: well-formed formulas (which have no 'kind') and terms (which have a kind depending on how many arguments they can take).[*]

A *well-formed expression* (wfe) defined recursively as follows.
(i) individual variables are well-formed expressions of kind 0; predicate variables are well-formed expressions both of kind 0 and kind *n* given by its

[*] Instead of 'kind', Cocchiarella speaks of meaningful expressions of differing 'types'. His type 1 is my "no kind", his type 0 is my "kind 0", and his type n+2 is my kind n+1.